Korean J. Math. **18** (2010), No. 4, pp. 387–394

ON THE GENERALIZED RANDERS CHANGE OF BERWALD METRICS

Nany Lee

Abstract. In this paper, we study the generalized Randers change ${}^*L(x,y) = L(x,y) + b_i(x,y)y^i$ from the Brewald metric L and the h-vector b_i. And in search for a non-Berwald Landsberg metric, we obtain the conditions on $b_i(x,y)$ under which ${}^*L$ is a Landsberg metric.

1. Introduction

Let (M, L) be an n-dimensional Finsler manifold with the Cartan connection

$$C\Gamma = (F_{j\,k}^{\,i}, G_j^i, C_{j\,k}^{\,i})\,.$$

L is called a Berwald metric if, in a standard local coordinate system (x^i, y^i) in $\widetilde{TM}$, $F_{j\,k}^{\,i}$ are functions of $x \in M$ only.

L is called a Landsberg metric if the Landsberg tensor

$$L_{j\,k}^{\,i} = \frac{\partial^2 G^i}{\partial y^j \partial y^k} - F_{j\,k}^{\,i}$$

vanishes, where $G^i = \frac{1}{2} F_{j\,k}^{\,i} y^j y^k$. By definition, every Berwald metric is a Landsberg metric.

The question whether there exists a non-Berwald Landsberg metric has been a long standing problem. In 2006, Z. Shen [12] showed that a regular (y-global) Landsberg (α, β) metric is a Berwald metric. In 2008, Z. Szabo [11] claimed that all regular Landsberg metrics are Berwald metrics. But V.S. Matveev[9] maintained that there are some mistakes in [11].

Received October 12, 2010. Revised November 5, 2010. Accepted November 11, 2010.

2000 Mathematics Subject Classification: 53C60, 58B20.

Key words and phrases: Finsler manifold, Berwald spaces, Randers change.

This work was supported by the University of Seoul 2009 Research Fund.

In 1995, R. Bryant [4] constructed an abundance of non-Berwald Landsberg metrics for 2-dimensional spaces in terms of a generalized Finsler metric. G.S. Asanov [1, 2] produced y-local (α, β) non-Berwald Landsberg metrics in dimension at least 3. The search for y-global non-Berwald Landsberg metrics should be continued.

In this paper, we will perturb the Berwald metric $L(x, y)$ in terms of the h-vector $b_i(x, y)$, and we will have a generalized Randers change

$$^*L(x, y) = L(x, y) + b_i(x, y)y^i$$

in order to try to get a non-Berwald Landsberg metric.

We wonder whether there exist some $b_i(x, y)$'s such that $^*L(x, y)$ is a non-Berwald Landsberg metric. Therefore, we will produce conditions under which $^*L(x, y)$ is a Landsberg metric.

In § 2, we set up the notations for Finsler manifolds and recall the Cartan connection on Finsler manifolds which can be characterized by the axioms. We will use the Cartan connection throughout this paper. Then we define the generalized Randers change of a Finsler metric and collect some results about the generalized Randers change.

In § 3, we will consider the generalized Randers change $^*L$ of Berwald metric L. For the computational simplicity, we assume that L is a Berwald metric. And we obtained the conditions under which $^*L$ is a Landsberg metric.

Acknowledgment . We would like to express deep gratitude to the referee for the very thorough reading of the paper and the helpful comments on the style. The referee also directed me to the paper by D. Bao [3].

2. Preliminaries

2.1. Finsler metric. Let M be a n-dimensional differentiable manifold with a local coordinate system $(x^1, \cdots, x^n)$.

And let $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ be the local coordinate system of the tangent bundle TM of M induced by $(x^1, \cdots, x^n)$.

A (real) Finsler metric L on the manifold M is a function $L : TM \to \mathbb{R}$ satisfying

(F1) L is smooth away from the zero section of TM,
(F2) $L(x, y) \geq 0$ and $L(x, y) = 0$ if and only if $y = 0$,
(F3) $L(x, \lambda y) = |\lambda| L(x, y)$ for all $\lambda \in \mathbb{R}$ and

(F4) L is strongly convex, i.e., $\left[\frac{\partial^2 L^2}{\partial y^i \partial y^j}\right]$ is positive definite.

Let (M, L) be an n-dimensional Finsler manifold with the fundamental function $L(x, y)$. Then the fundamental tensor, the angular metric tensor and the Cartan tensor are defined by

$$\begin{aligned} g_{ij}(x,y) &= \left(\frac{L^2}{2}\right)_{y^i y^j} \\ h_{ij}(x,y) &= L L_{y^i y^j} \\ C_{ijk}(x,y) &= \frac{1}{2}(g_{ij})_{y^k} = \left(\frac{L^2}{4}\right)_{y^i y^j y^k}. \end{aligned}$$

The indices on C are manipulated by g_{ij} and its inverse g^{ij}. We will use later

$$(g^{ij})_{y^k} = -2C^{i\,j}_{\ k}.$$

We consider the pull-back bundle $\tilde{\pi} : p^*TM \to \widetilde{TM}$ of the tangent bundle $\pi : TM \to M$ by the projection $p : \widetilde{TM} \to M$. Here $\widetilde{TM} = TM \setminus \{\text{zero section of } \pi : TM \to M\}$ is the slit tangent bundle.

$$\begin{array}{ccc} p^*TM & \xrightarrow{\ \tilde{p}\ } & TM \\ {\scriptstyle \tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \widetilde{TM} & \xrightarrow[\ p\ ]{} & M \end{array}$$

Then the strong convexity (F4) of L implies that the function g_{ij} on the slit tangent bundle $\widetilde{TM}$ of M defines a Riemannian structure on $\tilde{\pi} : p^*TM \to \widetilde{TM}$.

2.2. Cartan connection. As a generalization of Levi-Civita connection on the Riemannian manifold, we have Cartan connection on the Finsler manifold. Cartan connection $C\Gamma = (F^{\ i}_{j\ k}, G^i_j, C^{\ i}_{j\ k})$ on a Finsler manifold (M, L) can be characterized by the following axioms:

(C1) $C\Gamma$ is metrical,
(C2) the $(v)v$-torsion of $C\Gamma$ vanishes,
(C3) the $(h)h$-torsion of $C\Gamma$ vanishes,
(C4) the deflection tensor of $C\Gamma$ vanishes.

Note that the axiom (C4) determines a non-linear connection G_i^j of $\widetilde{TTM}$ uniquely. Then $\widetilde{TTM} = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is the horizontal subspace of $\widetilde{TTM}$ with the basis $\left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j} \right\}_{i=1}^n$ and $\mathcal{V}$ is the vertical subspace of $\widetilde{TTM}$ with the basis $\left\{ \frac{\partial}{\partial y^i} \right\}_{1=1}^n$. These subspaces $\mathcal{H}$ and $\mathcal{V}$ are identified with p^*TM by the isomorphisms $\chi^{\mathcal{H}} : p^*TM \to \mathcal{H}$ and $\chi^{\mathcal{V}} : p^*TM \to \mathcal{V}$ defined by $\chi^{\mathcal{H}}(\frac{\partial}{\partial x^i}) = \frac{\delta}{\delta x^i}$ and $\chi^{\mathcal{V}}(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$, respectively.

The h- and v-covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are given by

$$X_{i|j} = \partial_j X_i - (\dot{\partial}_h X_i) G_j^h - F_{i\,j}^{\,r}$$
$$X_{i \cdot j} = \dot{\partial}_j X_i - C_{i\,j}^{\,r} X_r$$

respectively, where $\partial_j = \frac{\partial}{\partial x^j}$ and $\dot{\partial}_j = \frac{\partial}{\partial y^j}$.

2.3. The generalized Randers change of Finsler metric. In 1980, H. Izumi [7] introduced the concept of an h-vector b_i, while studying the conformal transformation of Finsler spaces. The h-vector b_i is v-covariant constant with respect to the Cartan connection and satisfies

$$LC_{i\,j}^{\,h} b_h = \rho h_{ij}, \quad \rho \neq 0.$$

Then we show that

$$\dot{\partial}_j b_i = \frac{\rho h_{ij}}{L} = \rho L_{y^i y^j} \neq 0$$

and ρ is independent of directional arguments.

M. Matsumoto [8] introduced a transformation of Finsler metric given by

$${}^*L(x, y) = L(x, y) + b_i(x) y^i.$$

If L is a Riemannian metric, ${}^*L(x, y)$ is reduced to the Randers metric. So this transformation ${}^*L(x, y)$ is called the Randers change of Finsler metric.

Instead of the function b_i of coordinates x^i only, we will use the h-vector $b_i(x, y)$ and define the generalized Randers change

$${}^*L(x, y) = L(x, y) + b_i(x, y) y^i.$$

We can find some results regarding the generalized Randers change in B.N. Prasad [10] and M. Gupta and P. Pandey [6].

B.N. Prasad [10] obtained the relation between the Cartan connection coefficients $F^i_{j\,k}$ and ${}^*F^i_{j\,k}$ as

$$(2.1)\qquad {}^*F^i_{j\,k} = F^i_{j\,k} + D^i_{j\,k},$$

where

$$D^i_{j\,k} = LU^i_{j\,k} + \frac{1}{\tau}(U_{jk} - LU_{rjk}b^r)l^i,$$

$$U_{ijk} = \frac{L}{2}(L_{kjr}V^r_i - L_{ijr}V^r_k - L_{ikr}V^r_j) + L_{ij}A_k + L_{ik}A_j - L_{jk}A_i,$$

$$U_{jk} = E_{jk} - \frac{L}{2}(1+\rho)(L_{jr}V^r_k + L_{kr}V^r_j),$$

$$V_j = E_{j0} - F_{j0},$$

$$A_i = \frac{1}{2(1+\rho)}\rho_{|i} + \frac{1}{2\tau}(V_i - LV_{ri}b^r),$$

$$V_{ij} = \frac{1}{(1+\rho)}\left\{F_{ij} - LL_{ijk}F^k_0 + \frac{L_{ij}}{2^*L}\left\{(1+\rho)E_{00} - 2LF_{k0}b^k + {}^*L\rho_{|0}\right\}\right\}.$$

As usual,

$$U^{\,i}_{j\,k} = g^{li}U_{ljk}, \quad V^{\,i}_{j} = g^{il}V_{lj}, \quad F^{\,i}_{j} = g^{il}F_{lj},$$

and

$$2E_{ij} = b_{i|j} + b_{j|i}, \quad 2F_{ij} = b_{i|j} - b_{j|i}.$$

M. Gupta and P. Pandey [6] showed that if the h-vector b_i is a gradient, that is

$$b_{i|j} - b_{j|i} = 0 \text{ and } F_{ij} = 0,$$

then the generalized Randers change ${}^*L(x,y)$ becomes a projective change with projective factor $P = \frac{E_{00}}{2^*L}$ and the scalar $\rho(x)$ is constant.

Recall that ${}^*L$ is a projective change of L if and only if ${}^*L_{|k\cdot l}y^k ={}^* L_{\cdot l}$ and the projective factor $P(x,y) = \frac{{}^*L_{|k}y^k}{2^*L}$. For this, see [5].

Therefore, by direct calculation, we have a useful identities which will be used later.

Lemma 2.1. *If there exists a gradient h-vector $b_i(x,y)$ on TM, we have*

$$y^k\frac{\delta}{\delta x^k}(L_i) = \frac{\partial}{\partial x^j}L,$$

$$y^k\frac{\delta}{\delta x^k}L = 0.$$

3. Conditions under which $^*L$ is a Lansberg metric

In search for the non-Berwald Landsberg metric, we will assume that L is a Berwald metric and the h-vector b_i is a gradient. Let

$$^*L(x,y) = L(x,y) + b_i(x,y)y^i$$

be the generalized Randers change.

Note that if $^*L$ is a Berwald metric, $^*F^{\,i}_{j\,k}$ are functions of x only and by (2.1), which is equivalent to $\partial_j D^{\,i}_{m\,k} = 0$ for all i,j,k,m. And note that if $^*L$ is a Landsberg metric, the Landsberg tensor of $^*L$ vanishes. I.e.,

$$^*L^{\,i}_{j\,k} = y^m \frac{\partial D^{\,i}_{m\,k}}{\partial y^j} = 0 \quad \text{for all } i,j,k.$$

Now we have the necessary condition on $b_i(x,y)$ that $^*L$ is a Landsberg metric.

Theorem 3.1. *Let L be a Berwald metric on M and $b_i(x,y)$ be a gradient h-vector. If the generalized Randers change $^*L(x,y) = L(x,y)+ b_i(x,y)y^i$ is a Landsberg metric, then $b_i(x,y)$ satisfies*

$$L^i(\beta L_j - Lb_j) + L^2 L^i L_{jr} b^r - 2L^* L C^{ir}_j L_r = 0. \tag{3.1}$$

Proof. Since the Landsberg tensor of L vanishes, by (2.1) the Landsberg tensor of $^*L$

$$^*L^{\,i}_{j\,k} = L^{\,i}_{j\,k} + y^m \dot\partial D^{\,i}_{m\,k} = y^m \dot\partial D^{\,i}_{m\,k}$$

must vanish. I.e.,

$$y^m \dot\partial D^{\,i}_{m\,k} = 0$$

which is equivalent to

$$\dot\partial_j (D^{\,i}_{0\,k}) = D^{\,i}_{j\,k}.$$

And this implies that

$$\dot\partial_j (D^{\,i}_{0\,0}) = 2D^{\,i}_{j\,0},$$

using $D^{\,i}_{j\,k} = D^{\,i}_{k\,j}$.

On the other hand, B.N. Prasad [10] obtained the expression of $D^{\,i}_{0\,k}$:

$$D^{\,i}_{0\,k} = LV^i_k + \frac{1}{\tau}(V_k - LV_{rk}b^k)L^i.$$

And by direct calculation, we have

$$D^i_{00} = \frac{L^i}{\tau} E_{00}.$$

With the identities

$$\begin{aligned}
&\dot\partial_j\left(\frac{1}{\tau}\right) = \frac{\beta L_i - L b_j}{({}^*L)^2},\\
&\dot\partial_j(L^i) = L^i_j - 2C^{i\,r}_{\,j}L_r,\\
&\dot\partial_j(E_{mr})y^r = 0,\\
&\dot\partial_j(E_{00}) = E_{j0},
\end{aligned}$$

we get (3.1). □

Finally, we consider the necessary and sufficient condition that ${}^*L$ is a Landsberg metric. Note that ${}^*L$ is a Landsberg metric if and only if

$${}^*L^{\,i}_{j\,k} = y^m\dot\partial_j D^{\,i}_{m\,k} = 0$$

which is equivalent to

$$\dot\partial_j D^{\,i}_{0\,k} = D^{\,i}_{j\,k}.$$

Therefore, we have

Theorem 3.2. *For a Berwald metric L on M and a gradient h-vector $b_i(x,y)$, the generalized Randers change is a Landsberg metric if and only if $b_i(x,y)$ satisfies the differential equation*

$$\begin{aligned}
D^{\,i}_{j\,k} = {} & L_jV^i_k - 2LC^{i\,r}_{\,j}V_{rk} + \frac{1}{{}^*L}L^i(\beta L_j - Lb_j)(V_k - LV_{rk}b^k)\\
&+ \frac{1}{\tau}(L^i_j - 2C^{i\,r}_{\,j}L_r)(V_k - LV_{rk}b^r)\\
&+ \frac{L^i}{\tau}\left\{E_{jk} - L_jV_{rk}b^k - LV_{rk}(\rho L^r_j - 2C^{r\,s}_{\,j}b_s)\right\}\\
&+ \frac{1}{\tau}\left\{\frac{1}{2}L^{\,i}_{k\,j}E_{00} + L^i_kE_{j0} - (L_j + b_j)V^i_k\right\}\\
&- \frac{L^i}{\tau^2}\left\{\frac{1}{2}L_{rkj}E_{00} + L_{rk}E_{j0} - (L_j + b_j)V_{rk}\right\}b^r .
\end{aligned} \tag{3.2}$$

Remark . In order to find a non-Berwald Landsberg metric, it suffices to have $b_i(x,y)$ satisfying (3.2) and not all $\dot\partial_j(D^{\,i}_{m\,k}) = 0$.

References

[1] G.S. Asanov, *Finsleroid-Finsler spaces of positive-definite and relativistic types*, Rep. Math. Phys. **58**(2006), 275–300.

[2] G.S. Asanov, *Finsleroid-Finsler space and geodesic spray coefficient*, Publ. Math. Debrecen **71**(2007), 397–412.

[3] D. Bao, *On two curvature-driven problems in Riemann-Finsler geometry*, Adv. Stud. Pure Math. 48, Math. Soc. Japan. 19–71 Tokyo, 2007.

[4] R. Bryant, *Finsler structures on the* 2*-sphere satisfying* $K = 1$, Contemp. Math. **196**(1996), 27–41.

[5] S.S. Chern and Z. Shen, *Riemannian-Finsler geometry*, Nankai Tracts in Math. World Scientific, 2005.

[6] M.K. Gupta and P.N. Pandey, *On hypersurface of a Finsler space with a special metric*, Acta Math. Hungar. **120**(2008), 165–177.

[7] H. Izumi, *Conformal transformations Finsler spaces II. An h-conformally flat Finsler space*, Tensor (N.S.) **34**(1980), 337–359.

[8] M. Matsumoto, *On Finsler spaces with Randers metric and special forms of special tensors*, J. Math. Kyoto Univ. **14**(1974), 477–498.

[9] V.S. Matveev, *"All regular Landsber metris are Berwald" by Z. I. Szabo*, Preprint (2008).

[10] B.N. Prasad, *On the torsion tensors* R_{hjk} *and* P_{hjk} *of Finsler spaces with a metric* $ds = (g_{ij}(dx)dx^i dx^j)^1/2 + b_i(x,y)dx^i$, Indian J. Pure Appl. Math. **21**(1990), 27-39.

[11] Z. I. Szabo, *All regular Landsber metris are Berwald*, Ann. Global. Anal. Geom. **38**(2008), 381–386.

[12] Z. Shen, *On Landsberg* (α, β) *metrics*, Preprint (2006).

Department of Mathematics
The University of Seoul
Seoul, 130-743 Korea
E-mail: nany@uos.ac.kr