Korean J. Math. **18** (2010), No. 4, pp. 441–464

CONVERGENCE THEOREMS OF ITERATIVE ALGORITHMS FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES WITH APPLICATIONS

Shin Min Kang, Young Chel Kwun* and Xiaolong Qin

Abstract. In this paper, we introduce an iterative method for finding common elements of the set of solutions to a general system of variational inequalities for inverse-strongly accretive mappings and of the set of fixed points of strict pseudo-contractions in a real Banach space. The results presented in this paper mainly improve and extend the corresponding results announced by many others.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and P_C be the metric projection of H onto C. Recall that a mapping $A : C \to H$ is said to be *inverse-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C,$$

see [2], [7], [12], [27]. For such a case, A is said to be α-*inverse-strongly monotone*. Recall also that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Received November 13, 2010. Revised December 11, 2010. Accepted December 13, 2010.
2000 Mathematics Subject Classification: 49J30, 49J40, 47H05, 47H09.
Key words and phrases: sunny nonexpansive retraction, nonexpansive mapping, inverse-strongly accretive mapping, variational inequality.
This study was supported by research funds from Dong-A University.
* Corresponding author

T is said to be *strictly pseudo-contractive* [2] if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

It is easy to see that the class of strict pseudo-contractions includes the class of nonexpanive mappings as a special case. In this paper, we denote by $F(T)$ the set of fixed points of the mapping T.

Recall that the classical variational inequality problem, denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

For a given $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C$$

if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$.

One can see that the variational inequality problem (1.1) is equivalent to a fixed point problem. An element $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Let $A, B : C \to H$ be two nonlinear mappings. Next, we consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

which is said to be a *general system of variational inequalities*, where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (1.3) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.4}$$

which is defined by Verma [21]-[24]. The problem (1.4) is called the *new system of variational inequalities.* Further, if we add up the requirement that $x^* = y^*$, then problem (1.4) is reduced to the classical variational inequality (1.1). The problem of finding solutions of (1.3) and (1.4) by using iterative methods has been studied by many authors, see [6], [8], [11], [15], [21]-[24] and the references therein.

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of variational inequalities for α-inverse-strongly monotone mappings in the framework of real Hilbert space, see [6], [7], [12], [16], [27] and the references therein.

In 2007, Yao and Yao [27] introduced an iterative method for finding a common element of the set of fixed points of a single nonexpansive mapping and the set of solution of variational inequalities for a α-inverse-strongly monotone mapping. To be more precise, they proved the following theorem.

THEOREM 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \neq \emptyset$, where Ω denotes the set of solutions of a variational inequality for the α-inverse-strongly monotone mapping. Suppose that $\{x_n\}$ and $\{y_n\}$ are given by*

$$(1.5)\qquad \begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(I - \lambda_n A) y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2a]$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha_n$ and

(a) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap\Omega} u$.

Very recently, Ceng et al. [8] further improved the results of Yao and Yao [27] by considering the following iterative method:

$$
(1.6)\qquad \begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \quad \forall n \geq 1, \end{cases}
$$

where A, B are two different inverse-strongly monotone mappings. They obtained a strong convergence theorem by a relaxed extra-gradient method for the system of variational inequalities (1.3) and a nonexpansive mapping S.

In this paper, we improve the results of Ceng et al. [8] and Yao and Yao [27] from Hilbert spaces to Banach spaces. To be more precise, we consider an iterative method which involves a pair of inverse-strongly accretive mappings and a strict pseudo-contraction. Note that no Banach space is q-uniformly smooth for $q > 2$, see [25] for more details. We prove the strong convergence of the purposed iterative scheme in uniformly convex and 2-uniformly smooth Banach spaces. The results presented in this paper improve and extend the corresponding results announced by Ceng et al. [8], Iiduka and Takahashi [12], Yao and Yao [27] and many others.

2. Preliminaries

Let C be a nonempty closed convex subset of a Banach space E and E^* be the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$
J_q(x) = \{ f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \}, \quad \forall x \in E.
$$

In particular, $J = J_2$ is called the *normalized duality mapping.* It is known that $J_q(x) = \|x\|^{q-2} J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$, the identity mapping. Further, we have the following properties of the generalized duality mapping J_q:

(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in E$ with $x \neq 0$.
(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$.
(3) $J_q(-x) = -J_q(x)$ for all $x \in E$.

Let $U = \{x \in E : \|x\| = 1\}$. E is said to *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$. In this case, E is said to be *smooth.*

The *modulus of smoothness* of E is defined by

$$\rho(t) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| \leq t \right\}.$$

A Banach space E is said to be *uniformly smooth* if $\lim_{t \to 0} \frac{\rho(t)}{t} = 0$. Let $q > 1$. A Banach space E is said to be *q-uniformly smooth* if there exists a fixed constant $c > 0$ such that $\rho(t) \leq ct^q$. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

Note that

(1) E is a uniformly smooth Banach space if and only if J is single-valued and uniformly continuous on any bounded subset of E.

(2) All Hilbert spaces, L_p (or l_p) spaces ($p \geq 2$) and the Sobolev spaces W_m^p ($p \geq 2$) are 2-uniformly smooth, while L_p (or l_p) and W_m^p spaces ($1 < p \leq 2$) are p-uniformly smooth.

(3) Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p, where $p > 1$. More precisely, L_p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Next, we always assume that E is a smooth Banach space. Let C be a nonempty closed convex subset of E. Recall that an operator A of C into E is said to be *accretive* if

$$\langle Ax - Ay, J(x - y) \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be *α-inverse strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, J(x - y) \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

PROPOSITION 2.1. (Reich [17]) *Let E be a smooth Banach space and C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and J be the normalized duality mapping on E. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive.*
(b) *$\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$ for all $x, y \in E$.*
(c) *$\langle x - Qx, J(y - Qx)\rangle \leq 0$ for all $x \in E$ and $y \in C$.*

PROPOSITION 2.2. (Kitahara and Takahashi [13]) *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

For the class of nonexpansive mappings, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping ([2], [18]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. That is,

$$x_t = tu + (1 - t)Tx_t.$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [2] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [18] extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Reich [18] showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Proposition 2.3. *Let E be a uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D, that is, Q satisfies the property:*

$$\langle u - Qu, J(y - Qu)\rangle \leq 0, \quad \forall u \in C,\ y \in D.$$

Recently, Aoyama et al. [1] first considered the following generalized variational inequality problem in a smooth Banach space E.

Let C be a nonempty closed convex subset of E and A be an accretive operator of C into E. Find a point $u \in C$ such that

$$\langle Au, J(v - u)\rangle \geq 0, \quad \forall v \in C. \tag{2.2}$$

Next, we use $BVI(C, A)$ to denote the set of solutions of variational inequality problem (2.2).

Aoyama et al. [1] proved that the variational inequality problem (2.2) is equivalent to a fixed point problem. An element $u \in C$ is a solution of the variational inequality (2.2) if and only if $u \in C$ is a fixed point of the mapping $Q_C(I - \lambda A)$, where I is the identity mapping, $\lambda > 0$ is a constant and Q_C is a sunny nonexpansive retraction from E onto C, see [1] for more details.

Motivated by Aoyama et al. [1], we introduce the following general system of variational inequalities in a smooth Banach space.

Let $A : C \to E$ be a α-inverse strongly accretive mapping and $B : C \to E$ be a β-inverse strongly accretive mapping, respectively. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu B x^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{2.3}$$

where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then problem (2.3) is reduced to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu A x^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{2.4}$$

If we add up the requirement that $x^* = y^*$, then problem (2.4) is reduced to the classical variational inequality (2.2). In a real Hilbert space, the system (2.3) and (2.4) reduce to (1.3) and (1.4), respectively.

In order to prove our main results, we need the following lemmas and definitions.

Lemma 2.1. (Bruck [4]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$Sx = \sum_{n=1}^{\infty} \lambda_n T_n x, \quad \forall x \in C$$

is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

Lemma 2.2. (Suzuki [19]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and β_n be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

LEMMA 2.3. (Xu [26]) *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where γ_n *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

LEMMA 2.4. (Xu [25]) *Let* E *be a real* 2*-uniformly smooth Banach space with the best smooth constant* K*. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

LEMMA 2.5. (Browder [3]) *Let* E *be a uniformly convex Banach space and* C *be a nonempty closed convex subset of* E*. Let* $T : C \to C$ *be a nonexpansive mapping. Then* $I - T$ *is demi-closed at zero.*

LEMMA 2.6. *For given* $x^*, y^* \in C$*, where* $y^* = Q_C(x^* - \mu Bx^*)$*,* (x^*, y^*) *is a solution of problem* (2.3) *if and only if* x^* *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = Q_C[Q_C(x - \mu Bx) - \lambda A Q_C(x - \mu Bx)],$$

where Q_C *is a sunny nonexpansive retraction from* E *onto* C.

Proof.

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} x^* = Q_C(y^* - \lambda Ay^*), \\ y^* = Q_C(x^* - \mu Bx^*). \end{cases}$$

$$\Longleftrightarrow \quad x^* = Q_C[Q_C(x^* - \mu Bx^*) - \lambda A Q_C(x^* - \mu Bx^*)].$$

This completes the proof. □

Recall that $T : C \to C$ is said to be *strictly pseudo-contractive* if there exists a constant $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, J(x-y)\rangle$$
$$\leq \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C.$$

For the strict pseudo-contractions, we have the following result.

LEMMA 2.7. *Let C be a nonempty subset of a real 2-uniformly smooth Banach space E with the best smooth constant K. Let $T : C \to C$ be a strict pseudo-contraction with the constant $\lambda \in (0,1)$. For $a \in (0,1)$, define $T_\alpha x = (1-a)x + aTx$. Then, as $a \in (0,b)$, where $b = \min\left\{1, \frac{\lambda}{K^2}\right\}$ and T_a is a nonexpansive mapping with $F(T_a) = F(T)$.*

Proof. For any $x, y \in C$, from Lemma 2.4, one has

$$\begin{aligned}
&\|T_a x - T_a y\|^2 \\
&= \|(1-a)x + aTx - [(1-a)y + aTy]\|^2 \\
&= \|(1-y) + a[Tx - Ty - (x-y)]\|^2 \\
&\leq \|x-y\|^2 + 2a\langle Tx - Ty - (x-y), J(x-y)\rangle \\
&\quad + 2K^2a^2\|Tx - Ty - (x-y)\|^2 \\
&= \|x-y\|^2 + 2a\langle Tx - Ty, J(x-y)\rangle - 2a\|x-y\|^2 \\
&\quad + 2K^2a^2\|Tx - Ty - (x-y)\|^2 \\
&\leq \|x-y\|^2 + 2a[\|x-y\|^2 - \lambda\|Tx - Ty - (x-y)\|^2] \\
&\quad - 2a\|x-y\|^2 + 2K^2a^2\|Tx - Ty - (x-y)\|^2 \\
&= \|x-y\|^2 - 2a\lambda\|Tx - Ty - (x-y)\|^2 \\
&\quad + 2K^2a^2\|Tx - Ty - (x-y)\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

From the assumption, one obtains that T_a is nonexpansive. It is easy to see that $F(T_a) = F(T)$. □

3. Main results

Now, we are ready to give our main results.

THEOREM 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse-strongly accretive mapping and $B : C \to E$ be a β-inverse-strongly accretive* *mapping. Let $T : C \to C$ be a strict pseudo-contraction with the constant $\lambda \in (0, 1)$ such that $F(T) \neq \emptyset$. For any $x \in C$, define a mapping $S : C \to C$ by $Sx = (1 - a)x + aTx$, where $a \in \big(0, \min\big\{1, \frac{\lambda}{K^2}\big\}\big)$. Assume that $F := F(T) \cap F(G) \neq \emptyset$, where G is defined as Lemma 2.6. Suppose that $\{x_n\}$ is generated by*

$$\begin{cases} x_1 = u \in C, \\ y_n = Q_C(x_n - \mu Bx_n), \\ z_n = Q_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta_n Sx_n + (1 - \delta_n)z_n], \quad \forall n \geq 1, \end{cases} \tag{Δ}$$

where $\lambda \in \big(0, \frac{\alpha}{K^2}\big]$, $\mu \in \big(0, \frac{\beta}{K^2}\big]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ such that

(C1) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(C4) $\lim_{n\to\infty} \delta_n = \delta \in (0, 1)$.

Then the sequence $\{x_n\}$ defined by (Δ) converges strongly to $\bar{x} = Q_F u$ and $(\bar{x}, \bar{y})$ is a solution of problem (2.3), where $\bar{y} = Q_C(\bar{x} - \mu B\bar{x})$ and Q_F is a sunny nonexpansive retraction of C onto F.

Proof. From Lemma 2.7, we see that S is nonexpansive with $F(T) = F(S)$. It follows that $F(S)$ is closed and convex. Next, we show that the mappings $I - \lambda A$ and $I - \mu B$ are nonexpansive. Indeed, from the assumption $\lambda \in \big(0, \frac{\alpha}{K^2}\big]$ and Lemma 2.4, for all $x, y \in C$, we have

$$\begin{aligned} &\|(I - \lambda A)x - (I - \lambda A)y\|^2 \\ &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\langle Ax - Ay, J(x - y)\rangle + 2K^2\lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + 2K^2\lambda^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

This shows that $I - \lambda A$ is a nonexpansive mapping. In similar way, we can show that $I - \mu B$ is also nonexpansive. This implies that the mapping G is nonexpansive. It follows $F(G)$ is closed and convex.

This shows that $F = F(T) \cap F(G)$ is closed and convex. Letting $x^* \in F(T) \cap F(G)$, from Lemma 2.6, we have

$$x^* = Q_C[Q_C(x^* - \mu Bx^*) - \lambda AQ_C(x^* - \mu Bx^*)].$$

Putting $y^* = Q_C(x^* - \mu Bx^*)$, we obtain that

$$x^* = Q_C(y^* - \lambda Ay^*).$$

From the algorithm (Δ), we see

$$\begin{aligned} \|z_n - x^*\| &= \|Q_C(I - \lambda A)y_n - Q_C(I - \lambda A)y^*\| \\ &\leq \|(I - \lambda A)y_n - (I - \lambda A)y^*\| \\ &\leq \|y_n - y^*\| \\ &= \|Q_C(x_n - \mu Bx_n) - Q_C(x^* - \mu Bx^*)\| \\ &\leq \|(I - \mu B)x_n - (I - \mu B)x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.1}$$

Putting $t_n = \delta_n Sx_n + (1 - \delta_n)z_n$ for each $n \geq 1$, we obtain

$$\begin{aligned} \|t_n - x^*\| &\leq \delta_n\|Sx_n - x^*\| + (1 - \delta_n)\|z_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n u + \beta_n x_n + \gamma_n t_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|t_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\ &\leq \alpha_n\|u - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\} \\ &= \|u - x^*\|, \end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ all are bounded. On the other hand, we have

$$\begin{aligned} &\|t_{n+1} - t_n\| \\ &= \|\delta_{n+1}Sx_{n+1} + (1 - \delta_{n+1})z_{n+1} - [\delta_n Sx_n + (1 - \delta_n)z_n]\| \\ &\leq \delta_{n+1}\|Sx_{n+1} - Sx_n\| + (1 - \delta_{n+1})\|z_{n+1} - z_n\| \\ &\quad + \|Sx_n - z_n\||\delta_{n+1} - \delta_n| \\ &\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|M_1, \end{aligned} \tag{3.3}$$

where M_1 is an appropriate constant such that $M_1 \geq \sup_{n\geq 1}\{\|Sx_n - z_n\|\}$.

Next, we claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Putting $l_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$ for each $n \geq 1$, we see

$$x_{n+1} = (1-\beta_n)l_n + \beta_n x_n, \quad \forall n \geq 1. \tag{3.5}$$

Now, we compute $\|l_{n+1} - l_n\|$. From

$$\begin{aligned}
&l_{n+1} - l_n \\
&= \frac{\alpha_{n+1}u + \gamma_{n+1}t_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n t_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}u + \frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}t_{n+1} - \frac{\alpha_n}{1-\beta_n}u \\
&\quad - \frac{1-\beta_n-\alpha_n}{1-\beta_n}t_n \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u - t_{n+1}) + \frac{\alpha_n}{1-\beta_n}(t_n - u) + t_{n+1} - t_n,
\end{aligned}$$

we have

$$\begin{aligned}
\|l_{n+1} - l_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|t_n - u\| \\
&+ \|t_{n+1} - t_n\|.
\end{aligned} \tag{3.6}$$

Substituting (3.3) into (3.6), we arrive at

$$\begin{aligned}
&\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|t_n - u\| + |\delta_{n+1} - \delta_n| M_1.
\end{aligned}$$

It follows from the conditions (C2), (C3) and (C4) that

$$\lim_{n\to\infty} \|l_{n+1} - l_n\| - \|x_{n+1} - x_{n+1}\| < 0.$$

From Lemma 2.2, we obtain that

$$\lim_{n\to\infty} \|l_n - x_n\| = 0. \tag{3.7}$$

Thanks to (3.5), we see that

$$x_{n+1} - x_n = (1 - \beta_n)(l_n - x_n).$$

Combining the condition (C3) and (3.7), we obtain that (3.4) holds. Noting that

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(t_n - x_n),$$

and the conditions (C2) and (C3), we obtain

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{3.8}$$

Next, we show that

$$\limsup_{n\to\infty} \langle u - \bar{x}, J(x_n - \bar{x})\rangle \le 0, \tag{3.9}$$

where $\bar{x} = Q_F u$, Q_F is a sunny nonexpansive retraction of C onto F. Define a mapping $D : C \to C$ by

$$Dx = \delta Sx + (1 - \delta)Q_C(I - \lambda A)Q_C(I - \mu B)x, \quad \forall x, y \in C,$$

where $(0, 1) \ni \delta = \lim_{n\to\infty} \delta_n$. From Lemma 2.1, we see that the mapping D is a nonexpansive mapping such that

$$\begin{aligned} F(D) &= F(S) \cap F(Q_C(I - \lambda A)Q_C(I - \mu B)) \\ &= F(T) \cap F(G) = F. \end{aligned}$$

Note that

$$\begin{aligned} &\|t_n - Dx_n\| \\ &= \|\delta_n Sx_n + (1 - \delta_n)z_n - Dx_n\| \\ &= \|\delta_n Sx_n + (1 - \delta_n)z_n - \delta Sx_n \\ &\quad - (1 - \delta)Q_C(I - \lambda A)Q_C(I - \mu B)x_n\| \\ &\le |\delta_n - \delta| M_2, \end{aligned}$$

where M_2 is an appropriate constant such that

$$M_2 \geq \sup_{n\geq 1}\{\|Sx_n - (1-\delta)Q_C(I-\lambda A)Q_C(I-\mu B)x_n\|\}.$$

It follows that

$$\begin{aligned}
&\|x_n - Dx_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Dx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\|u - Dx_n\| + \beta_n\|x_n - Dx_n\| + \gamma_n\|t_n - Dx_n\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\|u - Dx_n\| + \beta_n\|x_n - Dx_n\| + \gamma_n|\delta_n - \delta|M_2.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1-\beta_n)\|x_n - Dx_n\| \leq \|x_n - x_{n+1}\| &+ \alpha_n\|u - Dx_n\| \\
&+ \gamma_n|\delta_n - \delta|M_2.
\end{aligned}$$

It follows from the conditions (C2)-(C4) and (3.4) that

$$\lim_{n\to\infty}\|x_n - Dx_n\| = 0.$$

Let z_t be the fixed point of the contraction $z \mapsto tu + (1-t)Dz$, where $t \in (0,1)$. That is,

$$z_t = tu + (1-t)Dz_t.$$

It follows that

$$\|z_t - x_n\| = \|(1-t)(Dz_t - x_n) + t(u - x_n)\|.$$

On the other hand, we have

$$\begin{aligned}
\|z_t - x_n\|^2 &\leq (1-t)^2\|Dz_t - x_n\|^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle \\
&\leq (1 - 2t + t^2)\|z_t - x_n\|^2 + f_n(t) \\
&\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2,
\end{aligned}$$

where

$$\begin{aligned}
f_n(t) &= (2\|z_t - x_n\| + \|x_n - Dx_n\|)\|x_n - Dx_n\| \\
&\to 0 \quad \text{as } n \to 0.
\end{aligned} \tag{3.10}$$

It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{3.11}$$

Letting $n \to \infty$ in (3.10) and noting (3.11), we arrive at

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \le \frac{t}{2}M, \tag{3.12}$$

where $M > 0$ is an appropriate constant such that $M \ge \|z_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 0$. Letting $t \to 0$ in (3.12), we have that

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \le 0.$$

So, for any $\epsilon > 0$, there exists a positive number δ_1, for $t \in (0,\delta_1)$, such that

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \le \frac{\epsilon}{2}. \tag{3.13}$$

On the other hand, we see that $Q_{F(D)}u = \lim_{t\to 0} z_t$ and $F(D) = F$. It follows that $z_t \to \bar{x} = Q_F u$ as $t \to 0$. There exists $\delta_2 > 0$ for $t \in (0,\delta_2)$, such that

$$\begin{aligned}
&|\langle u - \bar{x}, J(x_n - \bar{x})\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\
&\le |\langle u - \bar{x}, J(x_n - \bar{x})\rangle - \langle u - \bar{x}, J(x_n - z_t)\rangle| \\
&\quad + |\langle u - \bar{x}, J(x_n - z_t)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\
&\le |\langle u - \bar{x}, J(x_n - \bar{x}) - J(x_n - z_t)\rangle| + |\langle z_t - \bar{x}, J(x_n - z_t)\rangle| \\
&\le \|u - \bar{x}\|\|J(x_n - \bar{x}) - J(x_n - z_t)\| + \|z_t - \bar{x}\|\|x_n - z_t\| \\
&< \frac{\epsilon}{2}.
\end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}$ for all $t \in (0,\delta)$, we have that

$$\langle u - \bar{x}, J(x_n - \bar{x})\rangle \le \langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

This implies that

$$\limsup_{n\to\infty}\langle u - \bar{x}, J(x_n - \bar{x})\rangle \le \lim_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

It follows from (3.13) that

$$\limsup_{n\to\infty}\langle u-\bar{x}, J(x_n-\bar{x})\rangle \le \epsilon.$$

Since ϵ is chosen arbitrarily, we have that

$$\limsup_{n\to\infty}\langle u-\bar{x}, J(x_n-\bar{x})\rangle \le 0.$$

It follows from (3.4) and uniform smoothness of E that

$$\limsup_{n\to\infty}\langle u-\bar{x}, J(x_{n+1}-\bar{x})\rangle \le 0. \tag{3.14}$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Observe that

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2 \\
&= \langle \alpha_n u + \beta_n x_n + \gamma_n t_n - \bar{x}, J(x_{n+1}-\bar{x})\rangle \\
&= \alpha_n\langle u-\bar{x}, J(x_{n+1}-\bar{x})\rangle + \beta_n\langle x_n-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\
&\quad + \gamma_n\langle t_n-\bar{x}, J(x_{n+1}-\bar{x})\rangle \\
&\le \alpha_n\langle u-\bar{x}, J(x_{n+1}-\bar{x})\rangle + \beta_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\
&\quad + \gamma_n\|t_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\
&\le \alpha_n\langle u-\bar{x}, J(x_{n+1}-\bar{x})\rangle + (1-\alpha_n)\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\
&\le \alpha_n\langle u-\bar{x}, J(x_{n+1}-\bar{x})\rangle + \frac{1-\alpha_n}{2}(\|x_n-\bar{x}\|^2 + \|x_{n+1}-\bar{x}\|^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 \le{}& (1-\alpha_n)\|x_n-\bar{x}\|^2 \\
&+ 2\alpha_n\langle u-\bar{x}, J(x_{n+1}-\bar{x})\rangle.
\end{aligned} \tag{3.15}$$

From the condition (C2), (3.14) and applying Lemma 2.3 to (3.15), we obtain that

$$\lim_{n\to\infty}\|x_n-\bar{x}\| = 0.$$

This completes the proof. $\square$

Remark 3.1. Theorem 3.1 is applicable to L^p for all $p \geq 2$, however, we do not know whether it works in L^p for $1 < p < 2$.

From Lemma 2.1, we see that: For given $x^*, y^* \in C$, where $y^* = Q_C(x^* - \mu A x^*)$, (x^*, y^*) is a solution of problem (2.4) if and only if x^* is a fixed point of the mapping $H : C \to C$ defined by

$$H(x) = Q_C[Q_C(x - \mu Ax) - \lambda A Q_C(x - \mu Ax)], \tag{3.16}$$

where Q_C is a sunny nonexpansive retraction from E onto C.

If $A = B$, then Theorem 3.1 is reduced to the following.

Corollary 3.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and $A : C \to E$ be an α-inverse strongly-accretive mapping. Let $T : C \to C$ be a strict pseudo-contraction with the constant $\lambda \in (0, 1)$ such that $F(T) \neq \emptyset$. For any $x \in C$, define a mapping $S : C \to C$ by $Sx = (1 - a)x + aTx$, where $a \in \left(0, \min\left\{1, \frac{\lambda}{K^2}\right\}\right)$. Assume that $F := F(T) \cap F(H) \neq \emptyset$, where H is defined as (3.16). Suppose that $\{x_n\}$ is generated by*

$$\begin{cases} x_1 = u \in C, \\ y_n = Q_C(x_n - \mu A x_n), \\ z_n = Q_C(y_n - \lambda A y_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta S x_n + (1 - \delta) z_n], \quad \forall n \geq 1, \end{cases} \tag{3.17}$$

where $\delta \in (0, 1)$, $\lambda, \mu \in \left(0, \frac{\alpha}{K^2}\right]$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ such that

(C1) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ defined by (3.17) converges strongly to $\bar{x} = Q_F u$ and $(\bar{x}, \bar{y})$ is a solution of problem (2.4), where $\bar{y} = Q_C(\bar{x} - \mu A \bar{x})$ and Q_F is a sunny nonexpansive retraction of C onto F.

In a real Hilbert space, for the problem (1.3), we have the following theorem.

THEOREM 3.3. (Ceng et al. [8]) *For given* $x^*, y^* \in C$, *where* $y^* = P_C(x^* - \mu Bx^*)$, (x^*, y^*) *is a solution of problem* (1.3) *if and only if* x^* *is a fixed point of the mapping* $G' : C \to C$ *defined by*

$$G'(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)]. \tag{3.18}$$

It is well known that the smooth constant $K = \frac{\sqrt{2}}{2}$ in the framework of Hilbert spaces. From Theorem 3.1, we can obtain the following result immediately.

COROLLARY 3.4. *Let* H *be a real Hilbert space and* C *be a nonempty closed convex subset of* H. *Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping and* $B : C \to E$ *be a* β*-inverse-strongly monotone mapping. Let* $S : C \to C$ *be a nonexpansive mapping with a fixed point. Assume that* $F := F(S) \cap F(G') \neq \emptyset$, *where* G' *is defined as* (3.18). *Suppose that* $\{x_n\}$ *is generated by*

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \mu Bx_n), \\ z_n = P_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta Sx_n + (1-\delta)z_n], \quad \forall n \geq 1, \end{cases} \tag{3.19}$$

where $\delta \in (0,1)$, $\lambda \in (0, 2\alpha]$, $\mu \in (0, 2\beta]$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *such that*

(C1) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *defined by* (3.19) *converges strongly to* $\bar{x} = P_F u$ *and* $(\bar{x}, \bar{y})$ *is a solution of problem* (1.3), *where* $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

REMARK 3.2. If $f : C \to C$ is a contractive mapping and we replace u by $f(x_n)$ in the recursion formula (Δ), we can obtain the so-called viscosity iteration method. We note that all theorems and corollaries of this paper carry over trivially to the so-called viscosity iteration method, see [20] for more details.

4. Applications

In this section, we always assume that E is a uniformly convex and 2-uniformly smooth Banach space. Let C be a nonempty closed convex subset of E.

For the variational problem (2.2), In the case when $C = E$, we see that $BVI(E, A) = A^{-1}(0)$ holds, where

$$A^{-1}(0) = \{u \in E : Au = 0\}.$$

Taking $\lambda = \mu$, $C = E$ and $A = B$ in the problem (2.3), we have the following problem: Find $(x^*, y^*) \in E \times E$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, J(x - x^*)\rangle, & \forall x \in E, \\ \langle \lambda Ax^* + y^* - x^*, J(x - y^*)\rangle, & \forall x \in E. \end{cases} \tag{4.1}$$

From Lemma 2.6, we see that: For given $x^*, y^* \in E$, where $y^* = (x^* - \lambda Ax^*)$, (x^*, y^*) is a solution of the problem (4.1) if and only if x^* is a fixed point of the mapping $V : E \to E$ defined by

$$V(x) = (x - \lambda Ax) - \lambda A(x - \lambda Ax).$$

Therefore, we can obtain that $F(V) = A^{-1}(0)$. Indeed, it is sufficient to show that $F(V) \subseteq A^{-1}(0)$. Letting $x^* \in F(V)$, we see that

$$\begin{aligned} x^* &= (x^* - \lambda Ax^*) - \lambda A(x^* - \lambda Ax^*) \\ &= y^* - \lambda Ay^*, \end{aligned} \tag{4.2}$$

where

$$y^* = x^* - \lambda Ax^*. \tag{4.3}$$

Combining (4.2) with (4.3), we arrive at

$$2(x^* - y^*) = \lambda(Ax^* - Ay^*). \tag{4.4}$$

Suppose the contrary, $x^* \notin A^{-1}(0)$, i.e., $Ax^* \neq 0$. From (4.3), we see that $x^* \neq y^*$, which combines with (4.4) yields that $Ax^* \neq Ay^*$. Notice that

$$\begin{aligned} &\|x^* - y^*\|^2 \\ &= \|(y^* - \lambda Ay^*) - (x^* - \lambda Ax^*)\|^2 \\ &= \|(y^* - x^*) - \lambda(Ay^* - Ax^*)\|^2 \\ &\leq \|y^* - x^*\|^2 - 2\lambda\langle Ay^* - Ax^*, J(y^* - x^*)\rangle \\ &\quad + 2K^2\lambda^2\|Ay^* - Ax^*\|^2 \\ &\leq \|y^* - x^*\|^2 - 2\lambda\alpha\|Ay^* - Ax^*\|^2 + 2K^2\lambda^2\|Ay^* - Ax^*\|^2 \\ &= \|y^* - x^*\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ay^* - Ax^*\|^2 \\ &< \|y^* - x^*\|^2. \end{aligned}$$

This is a contradiction, which shows that $x^* \in A^{-1}(0)$. That is $F(V) = A^{-1}(0)$.

From above, we can obtain the following results immediately.

THEOREM 4.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and A be an α-inverse-strongly accretive mapping. Let T be a strict pseudo-contraction with the constant $\lambda \in (0,1)$ such that $F(T) \neq \emptyset$. For any $x \in E$, define a mapping $S : E \to E$ by $Sx = (1-a)x + aTx$, where $a \in \left(0, \min\left\{1, \frac{\lambda}{K^2}\right\}\right)$. Assume that $F := F(T) \cap A^{-1}(0) \neq \emptyset$. Suppose that $\{x_n\}$ is generated by*

$$
(4.5) \qquad \begin{cases} x_1 = u \in E, \\ y_n = x_n - \lambda A x_n, \\ z_n = y_n - \lambda A y_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta S x_n + (1-\delta) z_n], \quad \forall n \geq 1, \end{cases}
$$

where $\delta \in (0,1)$, $\lambda \in \left(0, \frac{\alpha}{K^2}\right]$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that

(C1) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ defined by (4.5) converges strongly to $\bar{x} = Q_F u$, where Q_F is a sunny nonexpansive retraction of E onto F.

It is well known that the class of pseudo-contractions is one of the most important classes of mappings among nonlinear mappings. To find a fixed point of pseudo-contraction is the central and important topics in nonlinear functional analysis. Within the past 40 years or so, mathematicians have been devoting to the studies on the existence and convergence of fixed points for pseudo-contractions. Closely related to the class of pseudo-contractive mappings is the class of accretive mappings. Recall that an operator A with domain $D(A)$ and range $R(A)$ in E is *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$,

$$\langle y_2 - y_1, J(x_2 - x_1)\rangle \geq 0.$$

An accretive operator A is *m-accretive* if $R(I + rA) = E$ for each $r > 0$. Next, we assume that A is m-accretive and has a zero (i.e., the inclusion $0 \in A(z)$ is solvable). The set of zeros of A is denoted by Ω. Hence,

$$\Omega = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = \Omega$ for all $r > 0$.

We observe that x is a zero of the accretive mapping A if and only if it is a fixed point of the pseudo-contractive mapping $T := I - A$. It is well known (see [9]) that if A is accretive then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts, especially within the past 15 years or so, have been devoted to iterative methods for approximating the zeros of accretive operators (see for example [5], [10], [14], [18], [28], [29]).

From Theorem 3.1, we can conclude the following result easily.

Theorem 4.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K. Let B be a β-inverse-strongly accretive mapping and A be a m-accretive mapping. Assume that $F := A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Suppose that $\{x_n\}$ is generated by*

$$(4.6) \qquad \begin{cases} x_1 = u \in E, \\ y_n = x_n - \mu B x_n, \\ z_n = y_n - \mu B y_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\delta J_r x_n + (1-\delta) z_n], \quad \forall n \geq 1, \end{cases}$$

where $\delta \in (0,1)$, $\mu \in \left(0, \frac{\beta}{K^2}\right]$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ such that

(C1) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$;

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ defined by (4.6) converges strongly to $\bar{x} = Q_F u$, where Q_F is a sunny nonexpansive retraction of E onto F.

References

[1] K. Aoyama, H. Iiduka and W. Takahashi, *Weak convergence of an iterative sequence for accretive operators in Banach spaces*, Fixed Point Theory Appl. **2006** (2006), Article ID 35390, 13 pages.

[2] F.E. Browder, *Fixed point theorems for noncompact mappings in Hilbert spaces*, Proc. Natl. Acad. Sci. USA **53** (1965), 1272–1276.

[3] F.E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. Sympos. Pure Math. **18** (1976), 78–81.

[4] R.E. Bruck, Jr., *Properties of fixed point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.

[5] R.E. Bruck, Jr., *A strongly convergent iterative method for the solution of* $0 \in Ux$ *for a maximal monotone operator* U *in Hilbert space*, J. Math. Anal. Appl. **48** (1974), 114–126.

[6] S.S. Chang, H.W.J. Lee and C.K. Chan, *Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces*, Appl. Math. Lett. **20** (2007), 329–334.

[7] L.C. Ceng and J.C. Yao, *An extragradient-like approximation method for variational inequality problems and fixed point problems*, Appl. Math. Comput. **190** (2007), 205–215.

[8] L.C. Ceng, C.Y. Wang and J.C. Yao, *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities*, Math. Methods Oper. Res. **67** (2008), 375–390.

[9] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publ., Dordrecht, 1990.

[10] C.E. Chidume and H. Zegeye, *Approximation methods for nonlinear operator equations*, Proc. Amer. Math. Soc. **131** (2003), 2467–2478.

[11] Z. Huang and M.A. Noor, *An explicit projection method for a system of nonlinear variational inequalities with different* (γ, r)*-cocoercive mappings*, Appl. Math. Comput. **190** (2007), 356–361.

[12] H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal. **61** (2005), 341–350.

[13] S. Kitahara and W. Takahashi, *Image recovery by convex combinations of sunny nonexpansive retractions*, Topol. Methods Nonlinear Anal. **2** (1993), 333–342.

[14] M.O. Osilike, *Iterative solution of nonlinear equations of the* Ψ*-strongly accretive type*, J. Math. Anal. Appl. **200** (1996), 259–271.

[15] X. Qin, S. M. Kang and M. Shang, *Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces*, Appl. Anal. **87** (2008), 421–430.

[16] X. Qin, M. Shang and H. Zhou, *Strong convergence of a general iterative method for variational inequality problems and fixed point problems in Hilbert spaces*, Appl. Math. Comput. **200** (2008), 242–253.

[17] S. Reich, *Asymptotic behavior of contractions in Banach spaces*, J. Math. Anal. Appl. **44** (1973), 57–70.

[18] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.
[19] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. **305** (2005), 227–239.
[20] T. Suzuki, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl. **325** (2007), 342–352.
[21] R.U. Verma, *On a new system of nonlinear variational inequalities and associated iterative algorithms*, Math. Sci. Res. Hot-Line **3** (1999), 65–68.
[22] R.U. Verma, *Generalized class of partial relaxed monotonicity and its connections*, Adv. Nonlinear Var. Inequal. **7** (2004), 155–164.
[23] R.U. Verma, *Generalized system for relaxed cocoercive variational inequalities and its projection methods*, J. Optim. Theory Appl. **121** (2004), 203–210.
[24] R.U. Verma, *General convergence analysis for two-step projection methods and applications to variational problems*, Appl. Math. Lett. **18** (2005), 1286–1292.
[25] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **1** (1991), 1127–1138.
[26] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. Lond. Math. Soc. **66** (2002), 240–256.
[27] Y. Yao and J.C. Yao, *On modified iterative method for nonexpansive mappings and monotone mappings*, Appl. Math. Comput. **186** (2007), 1551–1558.
[28] H. Zhou, *A characteristic condition for convergence of steepest descent approximation to accretive operator equations*, J. Math. Anal. Appl. **271** (2002), 1–6.
[29] H. Zhou, Y.J. Cho and S.M. Kang, *Characteristic conditions for convergence of generalized steepest descent approximation to multivalued accretive operator equations*, Comput. Math. Appl. **39** (2000), 1–11.

Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail: smkang@gnu.ac.kr

Department of Mathematics
Dong-A University
Pusan 614-714, Korea
E-mail: yckwun@dau.ac.kr

Department of Mathematics
Hangzhou Normal University
Hangzhou 310036, China
E-mail: ljjhqxl@yahoo.com.cn