Korean J. Math. **18** (2010), No. 4, pp. 465–472

# ON THE SIZE OF THE SET WHERE A MERMORPHIC FUNCTION IS LARGE

Ki-Ho Kwon

Abstract. In this paper, we investigate the extent of the set on which the modulus of a meromorphic function is lower bounded by a term related to some Nevanlinna Theory functionals. A. I. Shcherba estimate the size of the set on which the modulus of an entire function is lower bounded by 1. Our theorem in this paper shows that the same result holds in the case that the lower bound is replaced by $lT(r, f)$, $0 \le l < 1$ , which improves Shcherba's result. We also give a similar estimation for meromorphic functions.

## 1. Introduction and statements of results

Let $f$ be a nonconstant meromorphic function on $|z| \le \alpha r$ for some $\alpha > 1$ and $0 < r < \infty$. In this paper, for $0 \le l < 1$, we investigate the set

$$E(r, l, f) = \{\theta \in [0, 2\pi) : \log|f(re^{i\theta})| \ge l[m(r, f) - (1+\log\alpha)\, n(\alpha r, \infty, f)]\}$$

where $m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ and $n(t, \infty, f)$ denote the number of poles of $f$ in $|z| \le t$(see [6]). If $f$ is analytic in $|z| \le r$, then we rewrite

$$E(r, l, f) = \{\theta \in [0, 2\pi) : \log|f(re^{i\theta})| \ge lT(r, f)\}$$

where $T(r, f) = m(r, f)$ is the Nevanlinna characteristic of $f(z)$. In particular, if $f$ is analytic in $|z| \le r$ and $l = 0$, we denote

$$E(r, f) = E(r, 0, f) = \{\theta \in [0, 2\pi) : |f(re^{i\theta})| \ge 1\}.$$

Received November 16, 2010. Revised December 8, 2010. Accepted December 11, 2010.

2000 Mathematics Subject Classification: 30D35.

Key words and phrases: meromorphic function, lower bound, spread.

Supported by the Wharangdae Research Institute, Korea Military Academy, 2008.

Many results related to the lower bounds of $\limsup_{r\to\infty}|E(r,l,f)|$, called "the spread of $f$", can be found in [1], [2], [3] and [4] etc., where $|E|$ denotes the Lebesgue measure of the set $E$.

On the other hand, less attention have been given to obtain the lower bound for $\liminf_{r\to\infty}|E(r,l,f)|$. A. A. Gol'dberg [5] has constructed examples of entire functions $f(z)$ of order $\rho$, $\frac{1}{2}<\rho<\infty$, such that $\liminf_{r\to\infty}|E(r,f)|=0$.

In 1990, A. I. Shcherba established a sharp lower bound for the size of the set $E(r,f)$ in the following

Theorem 1.1 (Shcherba [8]). *Let $f(z)$ be a nonconstant entire function of a finite order $\rho$. Then*

$$\liminf_{r\to\infty}\frac{\log|E(r,f)|}{\log r}\geq -\frac{\rho}{2}.$$

This inequality is best possible in the following sense:

Theorem 1.2 (Shcherba [8]). *Let $\rho\in[0,\infty)$ be an arbitrary number. Then there exists an entire function $f(z)$ of order $\rho$ for which*

$$\liminf_{r\to\infty}\frac{\log|E(r,f)|}{\log r}=-\frac{\rho}{2}.$$

We improve Theorem 1.1 by proving

Theorem 1.3. *Let $f(z)$ be a nonconstant entire function of a finite order $\rho$. Then*

$$\liminf_{r\to\infty}\frac{\log|E(r,l,f)|}{\log r}\geq -\frac{\rho}{2}$$

*for all $l$, $0\leq l<1$.*

Note that if $l=0$ in Theorem 1.3, then the result of Theorem 1.3 is same as that of Theorem 1.1. Theorem 1.3 is easily deduced from the following

Theorem 1.4. *Let $f(z)$ be a nonconstant meromorphic function in $|z|\leq\alpha r$ for $1<\alpha<e^2$ and $0<r<\infty$. Suppose that*

$$m(r,f)-(1+\log\alpha)\,n(\alpha r,\infty,f)\geq 1.$$

*Then*

$$|E(r,l,f)|\geq\frac{2\pi(1-l)}{d_\alpha(1+\log 2)\sqrt{m(\alpha r,f)}+1-l}$$

*for all $l$, $0\leq l<1$, where $d_\alpha=\frac{4\sqrt{3\alpha}(\sqrt{\alpha}+1)}{\alpha-1}$.*

## 2. Proofs of Theorems

Let $f(z)$ be an analytic function in $|z| \le r$. Then the maximum modulus of $f(z)$ on $|z| = r$ is denoted by $M(r, f)$.

LEMMA 2.1. *Let $f(z)$ be a nonconstant analytic function in $|z| \le r$. Then*

$$|E(r, l, f)| \ge 2\pi \left[ \frac{1-l}{\frac{\log M(r,f)}{m(r,f)} - l} \right],$$

*for all $l$, $0 \le l < 1$.*

*Proof.* Suppose that $f(z)$ is a nonconstant analytic function in $|z| \le r$. Set $E = E(r, l, f)$ and $E^c = [0, 2\pi) - E$. Then

$$\begin{aligned} m(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \\ &= \frac{1}{2\pi} \Big[ \int_E \log^+ |f(re^{i\theta})| d\theta + \int_{E^c} \log^+ |f(re^{i\theta})| d\theta \Big] \\ &\le \frac{1}{2\pi} [|E| \log M(r, f) + (2\pi - |E|) l m(r, f)]. \end{aligned}$$

Hence

$$|E| \ge \frac{2\pi(1-l)m(r, f)}{\log M(r, f) - l m(r, f)} = 2\pi \left[ \frac{1-l}{\frac{\log M(r,f)}{m(r,f)} - l} \right].$$

□

LEMMA 2.2 (Kwon [7]). *Let $f(z)$ be a nonconstant analytic function in $|z| \le \alpha r$, $1 < \alpha < e^2$, and let $|f(0)| \ge 1$. Then*

$$\log M(r, f) \le d_\alpha \sqrt{m(r, f) m(\alpha r, f)}$$

*where $d_\alpha = \frac{4\sqrt{3\alpha}(\sqrt{\alpha}+1)}{\alpha - 1}$.*

LEMMA 2.3. *Let $\{b_n\}$ be the set of poles of a meromorphic function $f$, and let*

$$B(z) = \prod_{|b_n| \le R} \frac{R^2 - \overline{b_n} z}{R(z - b_n)}$$

*with $R = \alpha r$ and $z = re^{i\theta}$. Then we have*

$$m(r, B) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |B(re^{i\theta})| d\theta \le (1 + \log \alpha)\, n(\alpha r, \infty, f).$$

*Proof.* Let $b \in (0, R]$ be a real number and let $0 < \theta \leq \pi$. Then

$$\begin{aligned}
\left|\frac{R^2 - \overline{b}z}{R(z-b)}\right|^2 &= \left|\frac{(\alpha r - \frac{b}{\alpha}\cos\theta) - i\frac{b}{\alpha}\sin\theta}{(r\cos\theta - b) + ir\sin\theta}\right|^2 \\
&= \frac{(\alpha r - \frac{b}{\alpha}\cos\theta)^2 + (\frac{b}{\alpha}\sin\theta)^2}{(r\cos\theta - b)^2 + (r\sin\theta)^2} = \frac{(\alpha r)^2 + (\frac{b}{\alpha})^2 - 2br\cos\theta}{r^2 + b^2 - 2br\cos\theta} \\
&= \frac{(\alpha r + \frac{b}{\alpha})^2 - 2br(1+\cos\theta)}{(r+b)^2 - 2br(1+\cos\theta)} \leq \frac{(\alpha r + \frac{b}{\alpha})^2}{(r+b)^2 - 4br\cos^2\frac{\theta}{2}} \\
&\leq \frac{(\alpha r + \frac{b}{\alpha})^2}{(r+b)(r+b-2\sqrt{br}\cos\frac{\theta}{2})} \leq \frac{\alpha^2}{1 - \frac{2\sqrt{br}}{r+b}\cos\frac{\theta}{2}} \\
&\leq \frac{\alpha^2}{1-\cos\frac{\theta}{2}} = \frac{\alpha^2}{2\sin^2\frac{\theta}{4}}.
\end{aligned}$$

Hence we deduce that, for $0 < \theta \leq \pi$,

$$\left|\frac{R^2 - \overline{b}z}{R(z-b)}\right| \leq \frac{\alpha}{\sqrt{2}\sin\frac{\theta}{4}} \leq \frac{\alpha\pi}{\theta}.$$

Therefore we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{R^2 - \overline{b}z}{R(z-b)}\right| d\theta \leq \frac{1}{\pi}\int_0^{\pi} \log^+ \frac{\alpha\pi}{\theta} d\theta = 1 + \log\alpha.$$

If $b_n$ is a complex number satisfying $|b_n| = b$, then we claim that

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{R^2 - \overline{b_n}z}{R(z-b_n)}\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{R^2 - \overline{b}z}{R(z-b)}\right| d\theta.$$

In fact, if $b_n = be^{i\theta_0}$ and $z = re^{i\theta}$, then we have

$$\left|\frac{R^2 - \overline{b_n}z}{R(z-b_n)}\right| = \left|\frac{\alpha r - \frac{b}{\alpha}e^{-i\theta_0}e^{i\theta}}{re^{i\theta} - be^{i\theta_0}}\right| = \left|\frac{\alpha r - \frac{b}{\alpha}e^{i(\theta-\theta_0)}}{re^{i(\theta-\theta_0)} - b}\right|.$$

Recall that

$$\left|\frac{R^2 - \overline{b}z}{R(z-b)}\right| = \left|\frac{\alpha r - \frac{b}{\alpha}e^{i\theta}}{re^{i\theta} - b}\right|.$$

Hence our claim is proved by a change of variable in the integration. Thus we conclude that

$$m(r,B)=\sum_{|b_n|\le R}\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{R^2-\overline{b_n}z}{R(z-b_n)}\right|d\theta\le(1+\log\alpha)\,n(\alpha r,\infty,f).$$

□

*Proof of Theorem 1.4.* Let $f(z)$ be a nonconstant meromorphic function in $|z|\le\alpha r$ for $1<\alpha<e^2$. Suppose that

$$m(r,f)-(1+\log\alpha)\,n(\alpha r,\infty,f)\ge 1. \tag{1}$$

Let $b_n$ be the set of poles of $f$. Now we define functions $B(z)$ and $g(z)$ as

$$B(z)=\prod_{|b_n|\le R}\frac{R^2-\overline{b_n}z}{R(z-b_n)}$$

with $R=\alpha r$, and

$$g(z)=\frac{f(z)}{B(z)}.$$

Note that $g(z)$ is analytic in $|z|\le R$, and

$$m(r,g)\ge m(r,f)-(1+\log\alpha)\,n(\alpha r,\infty,f) \tag{2}$$

by Lemma 2.3. Hence it follows from (1), (2) and Lemma 2.1 that

$$m(r,g)\ge 1 \tag{3}$$

and

$$|E(r,l,g)|\ge 2\pi\left[\frac{1-l}{\frac{\log M(r,g)}{m(r,g)}-l}\right]. \tag{4}$$

If $|g(0)|\ge 1$, then we deduce from (4) and Lemma 2.2 that

$$|E(r,l,g)|\ge\frac{2\pi(1-l)}{d_\alpha\left[\frac{m(\alpha r,g)}{m(r,g)}\right]^{\frac{1}{2}}-l} \tag{5}$$

for all $l$, $0\le l<1$, where $d_\alpha=\frac{4\sqrt{3}\alpha(\sqrt{\alpha}+1)}{\alpha-1}$.

Now, suppose that $|g(0)|<1$. Then we can choose a number $c$ with $|c|\le 1$ such that $|g(0)+c|=1$. We set $h(z)=g(z)+c$, so that $h(z)$ satisfies all the hypotheses of Lemma 2.2. Therefore we get

$$\log M(r,h)\le d_\alpha\sqrt{m(r,h)m(\alpha r,h)}, \tag{6}$$

where $d_\alpha = \frac{4\sqrt{3\alpha}(\sqrt{\alpha}+1)}{\alpha-1}$. It follows from (3) and (6) that

$$\begin{aligned} \log M(r,g) - 1 &\le \log M(r, g+c) \le d_\alpha \sqrt{m(r,g+c)m(\alpha r, g+c)} \\ &\le d_\alpha \sqrt{[m(r,g)+\log 2][m(\alpha r, g)+\log 2]} \\ &\le d_\alpha \sqrt{(1+\log 2)^2 m(r,g) m(\alpha r, g)}, \end{aligned}$$

since $m(r,g) \ge 1$. Hence we have

$$\log M(r,g) \le d_\alpha (1+\log 2)\sqrt{m(r,g)m(\alpha r,g)} + 1. \tag{7}$$

It follows from (4) and (7) that

$$|E(r,l,g)| \ge \frac{2\pi(1-l)}{d_\alpha(1+\log 2)[\frac{m(\alpha r,g)}{m(r,g)}]^{1/2} + 1 - l}, \tag{8}$$

for all $l$, $0 \le l < 1$. By comparing (5) and (8), it is easy to see that (8) is always valid regardless of the value of $|g(0)|$.

In addition, since $|f(z)| = |g(z)|$ on $|z| = R$, and $|f(z)| > |g(z)|$ on $|z| = r < R$,

$$m(\alpha r, f) = m(\alpha r, g), \tag{9}$$

and

$$\{\theta \in [0, 2\pi) : \log|f(re^{i\theta}| \ge lm(r,g)\} \supseteq \{\theta \in [0, 2\pi) : \log|g(re^{i\theta}| \ge lm(r,g)\},$$

which implies that

$$|E(r,l,f)| \ge |E(r,l,g)|. \tag{10}$$

Thus we conclude from (3), (8), (9) and (10) that

$$|E(r,l,f)| \ge \frac{2\pi(1-l)}{d_\alpha(1+\log 2)\sqrt{m(\alpha r, f)} + 1 - l}$$

which proves the theorem. □

*Proof of Theorem 1.3.* If $f(z)$ is a polynomial, then the proof is trivial. Hence we assume that $f(z)$ is a transcendental entire function of order $\rho$. Then $f(z)$ satisfies all the hypotheses of Theorem 1.4, since $n(\alpha r, \infty, f) = 0$ and $m(r,f) \to \infty$ as $r \to \infty$. Thus we have

$$|E(r,l,f)| \ge \frac{2\pi(1-l)}{d_\alpha(1+\log 2)\sqrt{m(\alpha r, f)} + 1 - l}$$

for all $l$, $0 \le l < 1$.

Furthermore, if $\varepsilon > 0$ is given, then the definition of order gives

$$m(\alpha r, f) < r^{\rho+\varepsilon}$$

for all sufficiently large $r$. Therefore we finally get that

$$\liminf_{r\to\infty} \frac{\log|E(r,l,f)|}{\log r} \geq -\frac{\rho}{2},$$

which proves the theorem. □

## 3. Example

Let $f(z)$ be a meromorphic function in the complex plane of order $\rho$ and lower order $\lambda > 0$. Suppose that

$$\limsup_{r\to\infty} \frac{\log n(r,\infty,f)}{\log r} < \lambda. \tag{11}$$

Then we have

$$\liminf_{r\to\infty} \frac{\log|E(r,l,f)|}{\log r} \geq -\frac{\rho}{2}$$

for all $l$, $0 \leq l < 1$.

*Proof.* Let $f(z)$ have lower order $\lambda > 0$ and let its poles satisfy (11). Then we can choose $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ such that $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \lambda$ and

$$n(r,\infty,f) \leq r^{\lambda-\varepsilon_3}, \tag{12}$$

$$T(r,f) = m(r,f) + N(r,f) \geq r^{\lambda-\varepsilon_1},$$

and hence

$$m(r,f) \geq r^{\lambda-\varepsilon_2}, \tag{13}$$

for all sufficiently large $r$, since $N(r,f) \leq n(r,\infty,f)\log r$. Thus we deduce from (12) and (13) that, for given $\alpha > 1$,

$$m(r,f) - (1+\log\alpha)\, n(\alpha r,\infty,f) \geq 1$$

for all sufficiently large $r$. Therefore it follows from Theorem 1.3 that

$$\liminf_{r\to\infty} \frac{\log|E(r,l,f)|}{\log r} \geq -\frac{\rho}{2}$$

for all $l$, $0 \leq l < 1$. □

## References

[1] J. M. Anderson and A. Baernstein II, *The size of the set on which a meromorphic function is large*, Proc. Lond. Math. Soc., **36**(3)(1978), 518–539.
[2] K. Arima, *On maximum modulus of integral functions*, J. Math. Soc. Japan, **4**(1952), 62–66.
[3] A. Baernstein, *Proof of Edrei's conjecture*, Proc. Lond. Math. Soc., **25**(3)(1973), 418–434.
[4] A. Edrei, *Sums of deficiencies of meromorphic functions*, J. Anal. Math., **14**(1965) 79–107.
[5] A.A. Gol'dberg, *On the relationship between a defect and a deviations of a meromorphic function*, Teoriya Funktsii, Funkts. Analiz I Ikh Prilozh., **29**(1978), 31–35.
[6] W. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.
[7] K. Kwon, *On the growth of entire functions*, Israel J. Math., **86**(1994), 409–427.
[8] A.I. Shcherba, *On a set on which an entire function is lower bounded*, Russian Math. Izvestiya VUZ. Matematika, **34**(9)(1990), 75–81.

Department of Mathematics
Korea Military Academy
Seoul 139-799, Republic of Korea
*E-mail*: kkwon@kma.ac.kr