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NONTRIVIAL SOLUTIONS FOR THE NONLINEAR BIHARMONIC SYSTEM WITH DIRICHLET BOUNDARY CONDITION

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Abstract. We investigate the existence of multiple nontrivial solutions (ξ, η) for perturbations g_1, g_2 of the harmonic system with Dirichlet boundary condition

$$\Delta^2\xi + c\Delta\xi = g_1(2\xi + 3\eta) \quad \text{in } \Omega,$$
$$\Delta^2\eta + c\Delta\eta = g_2(2\xi + 3\eta) \quad \text{in } \Omega,$$

where we assume that $\lambda_1 < c < \lambda_2$, $g_1'(\infty)$, $g_2'(\infty)$ are finite. We prove that the system has at least three solutions under some condition on g.

1. Introduction

Let Ω be a smooth bounded region in R^n with smooth boundary $\partial\Omega$. Let $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ be the eigenvalues of $-\Delta$ with Dirichlet boundary condition in Ω. In [8] Jung and Choi studied the multiplicity of solutions of the nonlinear biharmonic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = g(u) \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

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where g is a differentiable function from R to R such that $g(0) = 0$, $c \in R$ and Δ^2 denotes the biharmonic operator. Here we assume that $g'(\infty) = \lim_{|u|\to\infty} \frac{g(u)}{u} \in R$.

In this paper we investigate the existence of multiple nontrivial solutions (ξ, η) for perturbations g_1, g_2 of the harmonic system with Dirichlet boundary condition

$$
\begin{aligned}
&\Delta^2\xi + c\Delta\xi = g_1(2\xi + 3\eta) \qquad \text{in } \Omega, \\
&\Delta^2\eta + c\Delta\eta = g_2(2\xi + 3\eta) \qquad \text{in } \Omega, \\
&\xi = 0, \eta = 0, \Delta\xi = 0, \Delta\eta = 0 \qquad \text{on } \partial\Omega,
\end{aligned} \tag{1.2}
$$

where we assume that $\lambda_1 < c < \lambda_2$, $g_1'(\infty)$, $g_2'(\infty)$ are finite.

Problem (1.1) was studied by Choi and Jung in [5], [6]. They showed that problem (1.1) has at least three solutions. The authors proved that (1.1) has at least two solutions by a variation of linking Theorem. The authors also proved in [7] that the problem

$$
\begin{aligned}
&\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \\
&u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega
\end{aligned} \tag{1.3}
$$

has at least two solutions by a variational reduction method when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ or $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$. This type problem arises in the study of travelling waves in a suspension bridge ([9,10,11]) or the study of the static deflection of an elastic plate in a fluid ([1,2,3,4,12,13]).

In section 2 we define a Banach space H spanned by eigenfunctions of $\Delta^2 + c\Delta$ with Dirichlet boundary condition. We recall a Linking Scale Theorem which will play a crucial role in our argument. In section 3 we prove that problem (1.1) has at least three solutions under some condition on g. In section 4 we investigate the existence of multiple nontrivial solutions (ξ, η) for perturbations g_1, g_2 of harmonic system (1.2).

2. Linking scale theorem

Let $\lambda_k (k = 1, 2, \ldots)$ denote the eigenvalues and $\phi_k (k = 1, 2, \ldots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, with the Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots, \lambda_i \to +\infty$ and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem $\Delta^2 u + c\Delta u = \mu u$ in Ω with the Dirichlet boundary condition $u = 0$, $\Delta u = 0$ on $\partial\Omega$, has infinitely many eigenvalues $\lambda_k(\lambda_k - c)$, $k = 1, 2, \ldots$, and corresponding eigenfunctions $\phi_k(x)$. The set of functions $\{\phi_k\}$ is an orthogonal base for $W_0^{1,2}(\Omega)$. Let us denote an element u of $W_0^{1,2}(\Omega)$ as $u = \sum h_k\phi_k, \sum h_k^2 < \infty$. Let c be not an eigenvalue of $-\Delta$ and define a subspace E of $W_0^{1,2}(\Omega)$ as follows

$$E = \{u \in W_0^{1,2}(\Omega) : \sum |\lambda_k(\lambda_k - c)|h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\sum |\lambda_k(\lambda_k - c)|h_k^2]^{\frac{1}{2}}.$$

We need the following some properties which are proved in [6, 7]. Since $\lambda_k \to +\infty$ and c is fixed, we have:
(i) $(\Delta^2 u + c\Delta)u \in E$ implies $u \in E$.
(ii) $|||u||| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $|||u||| = 0$.

Definition 2.1. *Let X be a Hilbert space, $Y \subset X$, $\rho > 0$ and $e \in X \backslash Y$, $e \neq 0$. Set:*

$$B_\rho(Y) = \{x \in Y | \|x\|_X \leq \rho\},$$

$$S_\rho(Y) = \{x \in Y | \|x\|_X = \rho\},$$

$$\Delta_\rho(e, Y) = \{\sigma e + v | \sigma \geq 0, v \in Y, \|\sigma e + v\|_X \leq \rho\},$$

$$\Sigma_\rho(e, Y) = \{\sigma e + v | \sigma \geq 0, v \in Y, \|\sigma e + v\|_X = \rho\} \cup \{v | v \in Y, \|v\|_X \leq \rho\}.$$

Now we recall a theorem of existence of three solutions which is linking scale theorem.

Theorem 2.1. *(Linking Scale Theorem) Let X be an Hilbert space, which is topological direct sum of the four subspaces X_0, X_1, X_2 and X_3. Let $F \in C^1(X, R)$. Moreover assume:*

(a) *$dim X_i < +\infty$ for $i = 0, 1, 2$;*
(b) *there exist $\rho > 0$, $R > 0$ and $e \in X_2$, $e \neq 0$ such that;*

$$\rho < R \qquad \text{and} \qquad \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e, X_3)} F;$$

(c) *there exist* $\rho' > 0$, $R' > 0$ *and* $e' \in X_1$, $e' \neq 0$ *such that:*

$$\rho' < R' \qquad \text{and} \qquad \sup_{S_{\rho'}(X_0 \oplus X_1)} F \leq \inf_{\Sigma_{R'}(e', X_2 \oplus X_3)} F;$$

(d) $R \leq R' (\Rightarrow \Delta_R(e, X_3) \subset \Sigma_{R'}(e', X_2 \oplus X_3))$;
(e) $-\infty < a = \inf_{\Delta_{R'}(e, X_2 \oplus X_3)} F$;
(f) $(P.S.)_c$ *holds for any* $c \in [a, b]$ *where* $b = \sup_{B_\rho(X_0 \oplus X_1 \oplus X_2)} F$.

Then there exist three critical levels c_1, c_2 *and* c_3 *for the functional* F *such that:*

$$a \leq c_3 \leq \sup_{S_{\rho'}(X_0 \oplus X_1)} F < \inf_{\Sigma_{R'}(e', X_2 \oplus X_3)} F \leq \inf_{\Delta_R(e, X_3)} F \leq c_2$$

$$\leq \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e.X_3)} F \leq c_1 \leq b.$$

Proposition 2.1. *Assume that* $g : E \to R$ *satisfies the assumptions of Theorem 1.1. Then all solutions in* $L^2(\Omega)$ *of*

$$\Delta^2 u + c\Delta u = g(u) \qquad \text{in } L^2(\Omega)$$

belong to E.

With the aid of Proposition 2.1 it is enough that we investigate the existence of solutions of (1.1) in the subspace E of $L^2(\Omega)$. Let $I : E \to R$ be the functional defined by,

$$I(u) = \int_\Omega \frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - G(u), \tag{2.1}$$

where $G(s) = \int_0^s g(\sigma)d\sigma$. Under the assumptions of Theorem 1.1, $I(u)$ is well defined. By the following Proposition, I is of class C^1 and the weak solutions of (1.1) coincide with the critical points of $I(u)$.

Proposition 2.2. *Assume that* $g(u)$ *satisfies the assumptions of Theorem 1.1. Then* $I(u)$ *is continuous and Frèchet differentiable in* E *and*

$$DI(u)(h) = \int_\Omega \Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - g(u)h \tag{2.2}$$

for $h \in X$. *Moreover* $\int_\Omega G(u)dx$ *is* C^1 *with respect to* u. *Thus* $I \in C^1$.

Let Z_2 act on E orthogonally. Then E has two invariant orthogonal subspaces Fix_{Z_2} and $Fix_{Z_2}^\perp$. Let us set

$$H = Fix_{Z_2}^\perp.$$

The Z_2 action has the representation $u \mapsto -u$, $\forall u \in H$. Thus Z_2 acts freely on the invariant subspace H. We note that H is a closed invariant linear subspace of E compactly embedded in $L^2(\Omega)$. It is easily checked that $\Delta^2 + c\Delta$ and g are equivariant on H, so I is invariant on H. Moreover $(\Delta^2 + c\Delta)(H) \subseteq H$, $\Delta^2 + c\Delta : H \to H$ is an isomorphism and $DI(H) \subseteq H$. Therefore critical points on H are critical points on E.

3. A single biharmonic equation

In this section we prove the existence of multiple solutions of the a nonlinear biharmonic equation.

Theorem 3.1. *Assume that* $\lambda_1 < c < \lambda_2$, $\lambda_k(\lambda_k - c) < g'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c)$, $\lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$ *and* $g'(t) \le \gamma < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, *where* $m \ge 1$, $k > 2$ *and* $\gamma \in R$. *Then problem (1.1) has at least three solutions.*

Let H_k be the subspace of H spanned by $\phi_1, \dots, \phi_k$ whose eigenvalues are $\lambda_1(\lambda_1 - c), \dots, \lambda_k(\lambda_k - c)$. Let $H_k^\perp$ be the orthogonal complement of H_k in H. Let $r = \frac{1}{2}\{\lambda_k(\lambda_k - c) + \lambda_{k+1}(\lambda_{k+1} - c)\}$ and let $L : H \to H$ be the linear continuous operator such that

$$(Lu, v) = \int_\Omega (\Delta^2 u + c\Delta u) \cdot v dx - r \int_\Omega uv dx.$$

Then L is symmetric, bijective and equivariant. The spaces H_k, $H_k^\perp$ are the negative space of L and the positive space of L. Moreover, there exists $\nu > 0$ such that

$$\forall u \in H_k \quad : \qquad (Lu, u) \le (\lambda_k(\lambda_k - r)) \int_\Omega u^2 dx \le -\nu \|\|u\|\|^2,$$

$$\forall u \in H_k^\perp \quad : \qquad (Lu, u) \ge (\lambda_{k+1}(\lambda_{k+1} - c)) \int_\Omega u^2 dx \ge \nu \|\|u\|\|^2.$$

We can write

$$I(u) = \frac{1}{2}(Lu, u) - \psi(u),$$

where

$$\psi(u) = \int_\Omega [G(u) - \frac{1}{2} r u^2] dx.$$

Since H is compactly embedded in L^2, the map $D\psi : X \to X$ is compact.

Lemma 3.1. *Assume that $g(u)$ satisfies the assumptions of Theorem 3.1. Then $I(u)$ satisfies the $(P.S.)_M$ condition for any $M \in R$.*

For the proof see [8].

Lemma 3.2. *Under the same assumptions of Theorem 3.1, The function $I(u)$ is bounded from above on H_k;*

$$\sup_{u \in H_k} I(u) < 0, \tag{3.1}$$

and from below on $H_k^\perp$; there exists $R_k > 0$ such that

$$\inf_{\substack{u \in H_k^\perp \\ |\|u\|| = R_k}} I(u) > 0 \tag{3.2}$$

and

$$\inf_{\substack{u \in H_k^\perp \\ |\|u\|| < R_k}} I(u) > -\infty. \tag{3.3}$$

Proof. For some constant $d \geq 0$, we have $G_r(s) \geq \frac{1}{2}\alpha s^2 + d$, where $G_r(s) = \int_0^s g_r(\sigma)d\sigma$. For $u \in H_k$,

$$\begin{aligned} (Lu, u) &\leq (\lambda_k(\lambda_k - c) - r) \int_\Omega u^2 dx \\ &= \frac{\lambda_k(\lambda_k - c) - \lambda_{k+1}(\lambda_{k+1} - c)}{2} \int_\Omega u^2, \end{aligned}$$

$$\int_\Omega G_r(u) \geq \frac{\alpha}{2} \int_\Omega u^2 + d|\Omega|,$$

so that

$$I(u) \leq \frac{1}{2} \cdot \frac{\lambda_k(\lambda_k - c) - \lambda_{k+1}(\lambda_{k+1} - c)}{2} \int_\Omega u^2 - \frac{\alpha}{2} \int_\Omega u^2 - d|\Omega| < 0,$$

since $\frac{\lambda_k(\lambda_k - c) - \lambda_{k+1}(\lambda_{k+1} - c)}{2} < \alpha$. Thus the functional I is bounded from above on H_k. Next we will prove that (3.2) and (3.3) hold. To get our claim (3.2), it is enough to prove that:

$$\lim_{\substack{u \in H^\perp \\ |\|u\|| \to +\infty}} I(u) = +\infty.$$

We have

$$\begin{aligned}
&\lim_{\substack{u\in H_k^\perp\\ |||u|||\to+\infty}} I(u)\\
&\geq \lim_{\substack{u\in H_k^\perp\\ |||u|||\to\infty}} \frac{1}{2}\Big(1-\frac{r}{\lambda_{k+1}(\lambda_{k+1}-c)}\Big)|||u|||^2 - \lim_{\substack{u\in H_k^\perp\\ |||u|||\to+\infty}} \int_\Omega G_r(u)dx\\
&\geq \lim_{\substack{u\in H_k^\perp\\ |||u|||\to+\infty}} \frac{1}{2}\Big(1-\frac{r}{\lambda_{k+1}(\lambda_{k+1}-c)}\Big)|||u|||^2 - \lim_{\substack{u\in H_k^\perp\\ |||u|||\to+\infty}} \frac{1}{2}\beta\int_\Omega u^2 - \bar{b}|\Omega|\\
&\geq \lim_{\substack{u\in H_k^\perp\\ |||u|||\to+\infty}} \frac{1}{2}\Big(1-\frac{r}{\lambda_{k+1}(\lambda_{k+1}-c)} - \frac{\beta}{\lambda_{k+1}(\lambda_{k+1}-c)}\Big)|||u|||^2 - \bar{b}|\Omega|\\
&\longrightarrow +\infty,
\end{aligned}$$

since there exists $\bar{b}\in R$ such that

$$G_r(u) < \frac{1}{2}\beta u^2 + \bar{b},$$

and

$$\beta < \frac{\lambda_{k+1}(\lambda_{k+1}-c) - \lambda_k(\lambda_k - c)}{2}.$$

Now we will prove (3.3). Since

$$\lambda_{k+m}(\lambda_{k+m}-c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$$

and

$$g'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1}-c),$$

there exists

$$\lambda_{k+m}(\lambda_{k+m}-c) < \bar{\gamma} < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$$

and $\bar{d}\geq 0$ such that $G(u) < \frac{\bar{\gamma}}{2}u^2 + \bar{d}$. Thus

$$\begin{aligned}
&\inf_{\substack{u\in H_k^\perp\\ |||u|||<R}} I(u)\\
&= \inf_{\substack{u\in H_k^\perp\\ |||u|||<R}} \{\frac{1}{2}|||u||| - \int_\Omega G(u)\}\\
&> \inf_{\substack{u\in H_k^\perp\\ |||u|||<R}} \{\frac{1}{2}\Big(1-\frac{\bar{\gamma}}{\lambda_{k+1}(\lambda_{k+1}-c)}\Big)|||u|||^2 - \bar{d}|\Omega|\} > -\infty.
\end{aligned}$$

□

LEMMA 3.3. *Under the same assumptions of Theorem 1.1, there exists $\rho_k > 0$ such that*

$$\sup_{\substack{u\in H_k\\ |||u|||=\rho_k}} I(u) < 0.$$

Proof. Let $L_\infty : H \to H$ be the linear operator defined by

$$(L_\infty u, v) = (\Delta^2 u + c\Delta u)v - g'(\infty)\int_\Omega uvdx,$$

where $\lambda_{i+1}(\lambda_{i+1} - c) < \lambda_k(\lambda_k - c) < g'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c)$, $k > i+1$. Then L_∞ is an isomorphism. The spaces H_k, and $H_k^\perp$ are the negative space of L_∞ and the positive space of L_∞ respectively, and

$$H = H_k \oplus H_k^\perp.$$

Set $G_\infty(s) = G(s) - \frac{1}{2}g'(\infty)s^2$. Then

$$I(u) = \frac{1}{2}(L_\infty u, u) - \int_\Omega G_\infty(s)dx.$$

Thus, by Lemma 4.2, $\lim_{\substack{u\in H\\ u\to 0}} \frac{1}{|||u|||^2}\int_\Omega G_\infty(u)dx \ge 0$. Then

$$\begin{aligned}\lim_{\substack{u\in H_k\\ u\to 0}} \frac{I(u)}{|||u|||^2} &< \lim_{\substack{u\in H_k\\ u\to 0}} \frac{1}{2|||u|||^2}[\lambda_k(\lambda_k - c) - g'(\infty)]\int_\Omega u^2\\ &- \lim_{\substack{u\in H_k\\ u\to 0}} \frac{1}{|||u|||^2}\int_\Omega G_\infty(u)dx < 0.\end{aligned}$$

thus there exists $\rho_k > 0$ such that

$$\sup_{\substack{u\in H_k\\ |||u|||=\rho_k}} < 0.$$

□

LEMMA 3.4. *Under the same assumptions of Theorem 1.1,*

$$\inf_{\substack{z\in H_k^\perp,\sigma\ge 0\\ |||z-\sigma e_1|||=R_k}} I(z - \sigma e_1) \ge 0.$$

Proof. By Lemma 3.2, there exists $R_k > 0$ such that

$$\inf_{\substack{u\in H_k^\perp\\ |||u|||=R_k}} I(u) > 0.$$

To get our claim, it is enough to prove that

$$\lim_{\substack{z\in H_k^{\perp},\sigma\geq 0,\\ |||z-\sigma e_1|||\to+\infty}} I(z-\sigma e_1)=+\infty. \tag{3.4}$$

To prove (3.4), we need to show that

$$\max_{\substack{z\in H_k^{\perp}\\ |||z|||=1}} \int z^2 = \max_{\substack{z\in H_k^{\perp},\sigma\geq 0,\\ |||z-\sigma e_1|||=1}} \int (z-\sigma e_1)^2. \tag{3.5}$$

In fact, we have immediately $\max_{\substack{z\in H_k^{\perp}\\ |||z|||=1}} \int z^2 \leq \max_{\substack{z\in H_k^{\perp},\sigma\geq 0\\ |||z-\sigma e_1|||=1}} \int (z-\sigma e_1)^2$. Now we prove that $\max_{\substack{z\in H_k^{\perp}\\ |||z|||=1}} \int z^2 \geq \max_{\substack{z\in H_k^{\perp},\sigma\geq 0\\ |||z-\sigma e_1|||=1}} \int (z-\sigma e_1)^2$.

If $\sigma>0$, then

$$2\int (z-\sigma e_1)v=\nu(z-\sigma e_1,v), \qquad \forall v\in H_1\oplus H_k^{\perp}.$$

Taking $v=z-\sigma e_1$ we get $\nu=2\int(z-\sigma e_1)^2$ and taking $v=e_1$ we also get

$$\begin{aligned} 0\leq 2\int (z-\sigma e_1)e_1 &= 2\int (z-\sigma e_1)^2(z-\sigma e_1,e_1)\\ &= -2\sigma\int (z-\sigma e_1)^2<0 \end{aligned}$$

which gives a contradiction. Then $z-\sigma e_1=z\in H_k^{\perp}$ and so $\max_{\substack{z\in H_k^{\perp}\\ |||z-\sigma e_1|||=1}} \int (z-\sigma e_1)^2=\max_{\substack{z\in H_k^{\perp}\\ |||z|||=1}} \int z^2$.

Thus we proved (3.5). Now we prove (3.4). For some constant β, $b\geq 0$, we have $G_\infty(s)\geq \frac{1}{2}\beta s^2+b$, where $G_\infty(s)=\int_0^s g_\infty(\sigma)d\sigma$, $g_\infty(s)=$

$g(s) - g'(\infty)s$. For $z \in H_k^{\perp}$ and $\sigma \geq 0$, by (4.5) we get

$$
\begin{aligned}
I \quad & (z - \sigma e_1) \\
\geq \quad & \frac{1}{2}|||z - \sigma e_1|||^2 - \frac{1}{2}g'(\infty)\int_{\Omega}(z - \sigma e_1)^2 - \frac{1}{2}\beta\int_{\Omega}(z - \sigma e_1)^2 - b|\Omega| \\
= \quad & \frac{1}{2}|||z - \sigma e_1|||^2(1 - g'(\infty)\int\frac{(z - \sigma e_1)^2}{|||z - \sigma e_1|||^2} - \beta\int\frac{(z - \sigma e_1)^2}{|||z - \sigma e_1|||^2}) - b|\Omega| \\
\geq \quad & \frac{1}{2}|||z - \sigma e_1|||^2(1 - (g'(\infty) + \beta)\max_{z \in H_k^{\perp}, \sigma \geq 0}\int\frac{(z - \sigma e_1)^2}{|||z - \sigma e_1|||^2}) - b|\Omega| \\
\geq \quad & \frac{1}{2}|||z - \sigma e_1|||^2(1 - (g'(\infty) + \beta)\max_{\substack{z \in H_k^{\perp} \\ |||z|||=1}}\int z^2) - b|\Omega| \longrightarrow \infty.
\end{aligned}
$$

as $|||z - \sigma e_1||| \to +\infty$. Thus we proved the lemma. □

From Lemma 3.3 and Lemma 3.4 we have

LEMMA 3.5. *Under the same assumptions of Theorem 1.1, there exists* $\rho_k > 0$ *such that*

$$\sup_{\substack{u \in H_k \\ |||u|||=\rho_k}} I(u) \leq \inf_{z \in \Sigma(-e_1, H_k^{\perp})} I(z - \sigma e_1),$$

where

$$
\begin{aligned}
& \Sigma(-e_1, H_k^{\perp}) \\
& = \{z \in H_k^{\perp} | |||z||| \leq R_k\} \cup \{z - \sigma e_1 | z \in H_k^{\perp}, \sigma \geq 0, |||z - \sigma e_1||| = R_k\},
\end{aligned}
$$

w4ith $R_k > \rho_k$.

LEMMA 3.6. *Let* $G_0 : R \to R$ *be a continuous function such that*

$$\inf_{s \in R}\frac{G_0(s)}{1 + s^2} > -\infty, \qquad \lim_{s \to 0}\frac{G_0(s)}{s^2} \geq 0.$$

Then

$$\lim_{\substack{u \to 0 \\ u \in H}}\frac{1}{|||u|||^2}\int_{\Omega}G_0(u)dx \geq 0.$$

Proof. Let

$$h(s) = \begin{cases} (\frac{G_o(s)}{s^2})^- & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

Then $h : R \to R$ is bounded, continuous, with $h(0) = 0$ and $G_0(s) \geq -h(s)s^2$. If (u_n) is a sequence in H with $u_n \to 0$, then up to a subsequence, $u_n \to 0$ a.e., and $v_n = \frac{u_n}{|||u_n|||}$ is strongly convergent in $L^2(\Omega)$. Since

$$\frac{1}{|||u_n|||^2}\int_\Omega G_0(u_n)dx \geq -\int_\Omega h(u_n)v_n^2 dx,$$

the claim follows. □

Lemma 3.7. *Under the same assumptions of Theorem 1.1, there exists* $\rho_{k+m} > 0$ *such that*

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m}}} I(u) < \inf_{z\in\Sigma(e_{k+m},H^{\perp}_{k+m})} I(z),$$

where $\Sigma(e_{k+m}, H^{\perp}_{k+m}) = \{w \in H^{\perp}_{k+m} | |||w||| \leq R_{k+m}\} \cup \{w + \sigma e_{k+m} | w \in H^{\perp}_{k+m}, \sigma \geq 0, |||w + \sigma e_{k+m}||| = R_{k+m}\}$ *with* $R_{k+m} > \rho_{k+m}.$

Proof. First we will prove that

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m},\rho\to 0}} I(u) < 0. \tag{3.6}$$

From the assumptions of Theorem 1.1, $\lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $m \geq 1$. Let $L_0 : H \to H$ be the linear operator defined by

$$(L_0u, v) = (\Delta^2 u + c\Delta u)v - g'(0)\int_\Omega uvdx.$$

Then L_0 is an isomorphism. The space H_{k+m}, $H^{\perp}_{k+m}$ are the negative space of L_0 and the positive space of L_0, respectively, and

$$H = H_{k+m} \oplus H^{\perp}_{k+m}.$$

Set $G_0(s) = G(s) - \frac{1}{2}g'(0)s^2$. Then

$$I(u) = \frac{1}{2}(L_0u, u) - \int_\Omega G_0(u)dx.$$

Note that $\inf \frac{G_0(s)}{1+s^2} > -\infty$, $\lim_{s\to 0} \frac{G_0(s)}{s^2} \geq 0$. Thus by Lemma 3.1, $\lim_{\substack{u\to 0\\ u\in H}} \frac{1}{|||u|||^2}\int_\Omega G_0(u)dx \geq 0$. Then

$$\begin{aligned}\lim_{\substack{u\to 0\\ u\in H_{k+m}}} \frac{I(u)}{|||u|||^2} &< \lim_{\substack{u\to 0\\ u\in H_{k+m}}} \frac{1}{2|||u|||^2}[\lambda_{k+m}(\lambda_{k+m}-c)-g'(0)]\int_\Omega u^2 \\ &- \lim_{\substack{u\to 0\\ u\in H_{k+m}}} \frac{1}{|||u|||^2}\int_\Omega G_0(u)dx < 0.\end{aligned}$$

Thus ther exists $\rho_{k+m} > 0$ such that $\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m},\rho\to 0}} I(u) < 0$. By Lemma 4.2, $\inf_{\substack{u\in H_k^\perp\\ |||u|||=R_k}} I(u) > 0$. Thus we have

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m},\rho_{k+m}\to 0}} I(u) < \inf_{\substack{u\in H_k^\perp\\ |||u|||=R_k}} I(u)$$

with $R_k > \rho_{k+m}$. In other words, there exists $e_{k+m} \in Span\{\phi_{k+1},\dots,\phi_{k+m}\}$ such that

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m},\rho_{k+m}\to 0}} I(u) < \inf_{\substack{u\in H_{k+m}^\perp\oplus e_{k+m}\\ e_{k+m}\in Span\{\phi_{k+1},\dots,\phi_{k+n}\},|||u|||=R_{k+m}}} I(u).$$

□

Proof of Theorem 3.1. By Lemma 3.5, there exists $\rho_k > 0$ such that

$$\sup_{\substack{u\in H_k\\ |||u|||=\rho_k}} I(u) \leq \inf_{z\in\Sigma(-e_1,H_k^\perp)} I(z-\sigma e_1),$$

where $\Sigma(-e_1,H_k^\perp) = \{z \in H_k^\perp | |||z||| \leq R_k\} \cup \{z-\sigma e_1 | z \in H_k^\perp, \sigma \geq 0, |||z-\sigma e_1||| = R_k\}$, with $R_k > \rho_k$. By Lemma 3.7, there exists $\rho_{k+m} > 0$ such that

$$\sup_{\substack{u\in H_{k+m}\\ |||u|||=\rho_{k+m}}} I(u) < \inf_{z\in\Sigma(e_{k+m},H_{k+m}^\perp)} I(z),$$

where $\Sigma(e_{k+m},H_{k+m}^\perp) = \{w \in H_{k+m}^\perp | |||w||| \leq R_{k+m}\} \cup \{w+\sigma e_{k+m} | w \in H_{k+m}^\perp, \sigma \geq 0, |||w+\sigma e_{k+m}||| = R_{k+m}\}$ with $R_{k+m} > \rho_{k+m}$ and $R_k > R_{k+m}$.Thus by Linking Scale Theorem 2.1., (1.1) has at least three solutions. □

4. Nontrivial solutions of biharmonic systems

In this section we investigate the existence of multiple nontrivial solutions (ξ,η) for perturbations g_1, g_2 of the harmonic system with Dirichlet boundary condition

$$\Delta^2\xi + c\Delta\xi = g_1(2\xi + 3\eta) \qquad \text{in } \Omega, \tag{4.1}$$

$$\Delta^2\eta + c\Delta\eta = g_2(2\xi + 3\eta) \qquad \text{in } \Omega,$$
$$\xi = 0, \eta = 0, \Delta\xi = 0, \Delta\eta = 0 \qquad\qquad \text{on } \partial\Omega,$$

where we assume that $\lambda_1 < c < \lambda_2$, $g_1'(\infty)$, $g_2'(\infty)$ are finite.

Theorem 4.1. *Assume that* $\lambda_1 < c < \lambda_2$,

$$\lambda_k(\lambda_k - c) < 2g_1'(\infty) + 3g_2'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c),$$

$$\lambda_{k+m}(\lambda_{k+m} - c) < 2g_1'(0) + 3g_2'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c).$$

Assume that $2g_1'(t)+3g_2'(t) \le \gamma < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$, *where* $m \ge 1$, $k > 2$ *and* $\gamma \in R$. *Then system (4.1) has at least three solutions.*

Proof. Let $L = \Delta^2 + c\Delta$. From problem (4.1) we get the equation

$$L(2\xi + 3\eta) = g(2\xi + 3\eta + 2) \qquad \text{in } \Omega, \tag{4.2}$$

$$\xi = 0, \eta = 0, \Delta\xi = 0, \Delta\eta = 0 \qquad\qquad \text{on } \partial\Omega,$$

where the nonlinearity $g(u) = 2g_1(u) + 3g_2(u)$.

Let $w = 2\xi + 3\eta$. Then the above equation is equivalent to

$$L(u) = g(u) \qquad \text{in } \Omega, \tag{4.3}$$

$$u = 0, \Delta u = 0 \qquad\qquad \text{on } \partial\Omega.$$

With the condition of the theorem, the above equation has at least three solutions , two of which are nontrivial solutions, say w_1, w_2. Hence we get the solutions (ξ,η) of problem (4.1) from the following systems:

$$L\xi = g_1(w_i) \qquad \text{in } \Omega, \tag{4.4}$$

$$L\eta = g_2(w_i) \qquad \text{in } \Omega,$$
$$\xi = 0, \eta = 0, \Delta\xi = 0, \Delta\eta = 0 \qquad\qquad \text{on } \partial\Omega,$$

where $i = 0, 1, 2$ and $w_0 = 0$. When $i = 0$, from the above equation we get the trivial solution $(\xi, \eta) = (0, 0)$. When $i = 1, 2$, from the above equation we get the nontrivial solutions (ξ_1, η_1), (ξ_2, η_2).

Therefore system(4.1) has at least three solutions (ξ, η), two of which are nontrivial solutions. □

THEOREM 4.2. *Assume that* $\lambda_1 < c < \lambda_2$,

$$2g_1'(\infty) + 3g_2'(\infty) < \lambda_1(\lambda_1 - c),$$

$$2g_1'(0) + 3g_2'(0) < \lambda_1(\lambda_1 - c).$$

Assume that $2g_1'(t) + 3g_2'(t) \leq \gamma < \lambda_1(\lambda_1 - c)$, *where* $\gamma \in R$. *Then system (4.1) has only the trivial solution* $(\xi, \eta) = (0, 0)$.

Proof. Let $L = \Delta^2 + c\Delta$. From problem (4.1) we get the equation

$$L(2\xi + 3\eta) = g(2\xi + 3\eta + 2) \qquad \text{in } \Omega, \tag{4.5}$$

$$\xi = 0, \eta = 0, \Delta\xi = 0, \Delta\eta = 0 \qquad \text{on } \partial\Omega,$$

where the nonlinearity $g(u) = 2g_1(u) + 3g_2(u)$.

Let $w = 2\xi + 3\eta$. Then the above equation is equivalent to

$$L(u) = g(u) \qquad \text{in } \Omega, \tag{4.6}$$

$$u = 0, \Delta u = 0 \qquad \text{on } \partial\Omega.$$

With the condition of the theorem, by Theorem 2.1 the above equation has the trivial solution. Hence we have the trivial solution $(\xi, \eta = (0, 0)$ of problem (4.1) from the following system:

$$L\xi = 0 \qquad \text{in } \Omega, \tag{4.7}$$

$$L\eta = 0 \qquad \text{in } \Omega,$$
$$\xi = 0, \eta = 0, \Delta\xi = 0, \Delta\eta = 0 \qquad \text{on } \partial\Omega.$$

From (4.7) we get the trivial solution $(\xi, \eta) = (0, 0)$. □

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