Korean J. Math. **18** (2010), No. 1, pp. 1–10

ALMOST HOMOMORPHISMS BETWEEN BANACH ALGEBRAS

Sung Jin Lee* and Choonkil Park

Abstract. It is shown that for an almost algebra homomorphism between Banach algebras, there exists a unique algebra homomorphism near the almost algebra homomorphism.

Moreover, we prove that for an almost algebra $*$-homomorphism between C^*-algebras, there exists a unique algebra $*$-homomorphism near the almost algebra $*$-homomorphism, and that for an almost algebra $*$-homomorphism between JB^*-algebras, there exists a unique algebra $*$-homomorphism near the almost algebra $*$-homomorphism

1. Introduction

Let E_1 and E_2 be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Rassias [5] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}||x||^p$$

for all $x \in E_1$. Găvruta [1] generalized the Rassias' result, and Park [4] applied the Găvruta's result to linear functional equations in Banach

Received September 24, 2009. Revised December 27, 2009. Accepted December 30, 2009.

2000 Mathematics Subject Classification: 47B48, 46K70, 39B52, 17C65, 46L05.

Key words and phrases: almost homomorphism, C^*-algebra, JB^*-algebra, stability, functional equation.

The first author was supported by the Daejin University Research Grant in 2010.

*Corresponding author.

modules over a C^*-algebra. In [2], the authors proved the stability of the functional equation $f(x+y+xy) = f(x)+f(y)+xf(y)+yf(x)$.

Throughout this paper, let $\mathcal{B}$ and $\mathcal{C}$ be complex Banach algebras with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

In this paper, we prove that for an almost algebra homomorphism $f:\mathcal{B}\to\mathcal{C}$, there exists a unique algebra homomorphism $h:\mathcal{B}\to\mathcal{C}$ near the almost algebra homomorphism. This result is applied to C^*-algebras and JB^*-algebras.

2. Stability of algebra homomorphisms between Banach algebras

We are going to show the generalized Hyers-Ulam stability of algebra homomorphisms between Banach algebras.

THEOREM 2.1. *Let $f:\mathcal{B}\to\mathcal{C}$ be a mapping with $f(0)=0$ for which there exists a function $\varphi:\mathcal{B}^4\to[0,\infty)$ such that*

$$\text{(i)}\qquad \widetilde{\varphi}(x,y,z,w) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty,$$

$$\begin{aligned}\|T_\mu f(x,y,z,w)\| :=& \|f(\mu x+\mu y+zw)-\mu f(x)-\mu f(y)-f(z)f(w)\|\\ \text{(ii)}\qquad & \leq \varphi(x,y,z,w)\end{aligned}$$

for all $\mu\in\mathbb{T}^1 := \{\lambda\in\mathbb{C} \mid |\lambda|=1\}$ and all $x,y,z,w\in\mathcal{B}$. Then there exists a unique algebra homomorphism $h:\mathcal{B}\to\mathcal{C}$ such that

$$\text{(iii)}\qquad \|f(x)-h(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x,x,0,0)$$

for all $x\in\mathcal{B}$.

Proof. Put $z=w=0$ and $\mu=1\in\mathbb{T}^1$ in (ii). Replacing y by x in (ii), we get

$$\|f(2x)-2f(x)\| \leq \varphi(x,x,0,0)$$

for all $x\in\mathcal{B}$. So one can obtain that

$$\|f(x)-\frac{1}{2}f(2x)\| \leq \frac{1}{2}\varphi(x,x,0,0),$$

and hence

$$\|\frac{1}{2^n}f(2^n x)-\frac{1}{2^{n+1}}f(2^{n+1}x)\| \leq \frac{1}{2^{n+1}}\varphi(2^n x,2^n x,0,0)$$

for all $x \in \mathcal{B}$. So we get

$$\|f(x) - \frac{1}{2^n} f(2^n x)\| \leq \frac{1}{2} \sum_{l=0}^{n-1} \frac{1}{2^l} \varphi(2^l x, 2^l x, 0, 0) \tag{1}$$

for all $x \in \mathcal{B}$.

Let x be an element in $\mathcal{B}$. For positive integers n and m with $n > m$,

$$\|\frac{1}{2^n} f(2^n x) - \frac{1}{2^m} f(2^m x)\| \leq \frac{1}{2} \sum_{l=m}^{n-1} \frac{1}{2^l} \varphi(2^l x, 2^l x, 0, 0),$$

which tends to zero as $m \to \infty$ by (i). So $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{C}$ is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $h : \mathcal{B} \to \mathcal{B}$ by

$$h(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \tag{2}$$

for all $x \in \mathcal{B}$.

By (i) and (2), we get

$$\|T_1 h(x, y, 0, 0)\| = \lim_{n \to \infty} \frac{1}{2^n} \|T_1 f(2^n x, 2^n y, 0, 0)\| \leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 0, 0) = 0$$

for all $x, y \in \mathcal{B}$. Hence $T_1 h(x, y, 0, 0) = 0$ for all $x, y \in \mathcal{B}$. So one can obtain that h is additive. Moreover, by passing to the limit in (1) as $n \to \infty$, we get the inequality (iii).

Now let $S : \mathcal{B} \to \mathcal{C}$ be another additive mapping satisfying

$$\|f(x) - S(x)\| \leq \frac{1}{2} \widetilde{\varphi}(x, x, 0, 0)$$

for all $x \in \mathcal{B}$.

$$\begin{aligned} \|h(x) - S(x)\| &= \frac{1}{2^l} \|h(2^l x) - S(2^l x)\| \\ &\leq \frac{1}{2^l} \|h(2^l x) - f(2^l x)\| + \frac{1}{2^l} \|f(2^l x) - S(2^l x)\| \\ &\leq \frac{2}{2} \frac{1}{2^l} \widetilde{\varphi}(2^l x, 2^l x, 0, 0), \end{aligned}$$

which tends to zero as $l \to \infty$ by (i). Thus $h(x) = S(x)$ for all $x \in \mathcal{B}$. This proves the uniqueness of h.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\|f(2^n \mu x) - 2\mu f(2^{n-1} x)\| \leq \varphi(2^{n-1} x, 2^{n-1} x, 0, 0)$$

for all $x \in \mathcal{B}$. And one can show that

$$\|\mu f(2^n x) - 2\mu f(2^{n-1}x)\| \le \varphi(2^{n-1}x, 2^{n-1}x, 0, 0)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. So

$$\begin{aligned}\|f(2^n \mu x) - \mu f(2^n x)\| \le& \|f(2^n \mu x) - 2\mu f(2^{n-1}x)\| + \|2\mu f(2^{n-1}x) - \mu f(2^n x)\| \\ \le& 2\varphi(2^{n-1}x, 2^{n-1}x, 0, 0)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. Thus $2^{-n}\|f(2^n \mu x) - \mu f(2^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. Hence

$$h(\mu x) = \lim_{n\to\infty} \frac{f(2^n \mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu f(2^n x)}{2^n} = \mu h(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [3, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. And $h(x) = h(3 \cdot \frac{1}{3}x) = 3h(\frac{1}{3}x)$ for all $x \in \mathcal{B}$. So $h(\frac{1}{3}x) = \frac{1}{3}h(x)$ for all $x \in \mathcal{B}$. Thus

$$\begin{aligned}h(\lambda x) &= h(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot h(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}h(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}h(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(h(\mu_1 x) + h(\mu_2 x) + h(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)h(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}h(x) \\ &= \lambda h(x)\end{aligned}$$

for all $x \in \mathcal{B}$. Hence

$$h(\alpha x + \beta y) = h(\alpha x) + h(\beta y) = \alpha h(x) + \beta h(y)$$

for all $\alpha, \beta \in \mathbb{C}(\alpha, \beta \neq 0)$ and all $x, y \in \mathcal{B}$. And $h(0x) = 0 = 0h(x)$ for all $x \in \mathcal{B}$. So the unique additive mapping $h : \mathcal{B} \to \mathcal{C}$ is a $\mathbb{C}$-linear mapping.

It follows from (2) that

$$h(x) = \lim_{n\to\infty} \frac{f(2^{2n}x)}{2^{2n}} \tag{3}$$

for all $x \in \mathcal{B}$. Let $x = y = 0$ in (ii). Then we get

$$\|f(zw) - f(z)f(w)\| \le \varphi(0, 0, z, w)$$

for all $z, w \in \mathcal{B}$. Since

$$\frac{1}{2^{2n}}\varphi(0,0,2^n z, 2^n w) \le \frac{1}{2^n}\varphi(0,0,2^n z, 2^n w),$$

$$\begin{aligned}\frac{1}{2^{2n}}\|f(2^n z\cdot 2^n w) - f(2^n z)f(2^n w)\| &\le \frac{1}{2^{2n}}\varphi(0,0,2^n z,2^n w)\\ &\le \frac{1}{2^n}\varphi(0,0,2^n z,2^n w)\end{aligned} \tag{4}$$

for all $z, w \in \mathcal{B}$. By (i), (3), and (4),

$$\begin{aligned}h(zw) &= \lim_{n\to\infty}\frac{f(2^{2n}zw)}{2^{2n}} = \lim_{n\to\infty}\frac{f(2^n z\cdot 2^n w)}{2^n\cdot 2^n}\\ &= \lim_{n\to\infty}(\frac{f(2^n z)}{2^n}\cdot\frac{f(2^n w)}{2^n}) = \lim_{n\to\infty}\frac{f(2^n z)}{2^n}\cdot\lim_{n\to\infty}\frac{f(2^n w)}{2^n}\\ &= h(z)h(w)\end{aligned}$$

for all $z, w \in \mathcal{B}$. Hence the additive mapping $h : \mathcal{B} \to \mathcal{C}$ is an algebra homomorphism satisfying the inequality (iii), as desired. □

Corollary 2.2. *Let $f : \mathcal{B} \to \mathcal{C}$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0,1)$ such that*

$$\|f(\mu x+\mu y+zw)-\mu f(x)-\mu f(y)-f(z)f(w)\| \le \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, w \in \mathcal{B}$. Then there exists a unique algebra homomorphism $h : \mathcal{B} \to \mathcal{C}$ such that

$$\|f(x) - h(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p$$

for all $x \in \mathcal{B}$.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$, and apply Theorem 2.1. □

Theorem 2.3. *Let $f : \mathcal{B} \to \mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B}^4 \to [0,\infty)$ satisfying* (i) *such that*

(iv) $\|f(\mu x + \mu y + zw) - \mu f(x) - \mu f(y) - f(z)f(w)\| \le \varphi(x, y, z, w)$

for $\mu = 1, i$, and all $x, y, z, w \in \mathcal{B}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$, then there exists a unique algebra homomorphism $h : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iii).

Proof. Put $z = w = 0$ and $\mu = 1$ in (iv). By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $h : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iii). By the same reasoning as the proof of [5, Theorem], the additive mapping $h : \mathcal{B} \to \mathcal{C}$ is $\mathbb{R}$-linear.

Put $z = w = 0$ and $\mu = i$ in (iv). By the same method as the proof of Theorem 2.1, one can obtain that

$$h(ix) = \lim_{n\to\infty} \frac{f(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{if(2^n x)}{2^n} = ih(x)$$

for all $x \in \mathcal{B}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = \eta + i\nu$, where $\eta, \nu \in \mathbb{R}$. So

$$\begin{aligned} h(\lambda x) &= h(\eta x + i\nu x) = \eta h(x) + \nu h(ix) = \eta h(x) + i\nu h(x) \\ &= \lambda h(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{B}$. So

$$h(\alpha x + \beta y) = h(\alpha x) + h(\beta y) = \alpha h(x) + \beta h(y)$$

for all $\alpha, \beta \in \mathbb{C}$, and all $x, y \in \mathcal{B}$. Hence the additive mapping $h : \mathcal{B} \to \mathcal{C}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

3. Stability of algebra $*$-homomorphisms between C^*-algebras

In this section, let $\mathcal{B}$ be a unital C^*-algebra with unitary group $\mathcal{U}(\mathcal{B})$, and $\mathcal{C}$ a C^*-algebra.

We are going to show the generalized Hyers-Ulam-Rassias stability of algebra $*$-homomorphisms between C^*-algebras.

THEOREM 3.1. *Let $f : \mathcal{B} \to \mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{B}^4 \to [0, \infty)$ satisfying* (i) *and* (ii) *such that*

$$\text{(v)} \qquad \|f(2^n u^*) - f(2^n u)^*\| \le \varphi(2^n u, 2^n u, 0, 0)$$

for all $u \in \mathcal{U}(\mathcal{B})$ and $n = 0, 1, \cdots$. Then there exists a unique algebra $*$-homomorphism $h : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iii).

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique algebra homomorphism $h : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iii).

It follows from (i) and (v) that

$$h(u^*) = \lim_{n\to\infty} \frac{f(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{f((2^n u)^*)}{2^n} = \lim_{n\to\infty} \frac{f(2^n u)^*}{2^n} = (\lim_{n\to\infty} \frac{f(2^n u)}{2^n})^*$$
$$= h(u)^*$$

for all $u \in \mathcal{U}(\mathcal{B})$.

Now let $x \in \mathcal{B}$ ($x \neq 0$) and M an integer greater than $4\|x\|$. Then $\|\frac{x}{M}\| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [3, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(\mathcal{B})$ such that $3\frac{x}{M} = u_1 + u_2 + u_3$. So

$$\begin{aligned} h(x^*) &= h(\frac{M}{3}(u_1^* + u_2^* + u_3^*)) = \frac{M}{3} h(u_1^* + u_2^* + u_3^*) \\ &= \frac{M}{3}(h(u_1^*) + h(u_2^*) + h(u_3^*)) \\ &= \frac{M}{3}(h(u_1)^* + h(u_2)^* + h(u_3)^*) \\ &= \frac{M}{3}(h(u_1) + h(u_2) + h(u_3))^* \\ &= \frac{M}{3}(h(u_1 + u_2 + u_3))^* = h(\frac{M}{3}(u_1 + u_2 + u_3))^* \\ &= h(x)^* \end{aligned}$$

for all $x \in \mathcal{B}$. Hence the algebra homomorphism $h : \mathcal{B} \to \mathcal{C}$ is involutive, as desired. □

Corollary 3.2. *Let $f : \mathcal{B} \to \mathcal{C}$ be a mapping with $f(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|f(\mu x + \mu y + zw) - &\mu f(x) - \mu f(y) - f(z) f(w)\| \\ &\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \\ \|f(2^n u^*) - f(2^n u)^*\| &\leq 2 \cdot 2^{np}\theta \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{B})$, $n = 0, 1, \cdots$, and all $x, y, z, w \in \mathcal{B}$. Then there exists a unique algebra $*$-homomorphism $h : \mathcal{B} \to \mathcal{C}$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in \mathcal{B}$.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$, and apply Theorem 3.1. □

THEOREM 3.3. *Let* $f : \mathcal{B} \to \mathcal{C}$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi : \mathcal{B}^4 \to [0, \infty)$ *satisfying* (i), (iv), *and* (v). *If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{B}$, *then there exists a unique algebra* $*$*-homomorphism* $h : \mathcal{B} \to \mathcal{C}$ *satisfying the inequality* (iii).

Proof. By the same reasoning as in the proofs of Theorems 2.1 and 2.3, there exists a unique algebra homomorphism $h : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iii).

By the same method as in the proof of Theorem 3.1, one can show that the algebra homomorphism $h : \mathcal{B} \to \mathcal{C}$ is involutive, as desired. □

4. Stability of algebra $*$-homomorphisms between JB^*-algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [6]). Let $\mathcal{H}$ be a complex Hilbert space, regarded as the "state space" of a quantum mechanical system. Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ (not necessarily given by an anticommutator) is called a *Jordan algebra* if $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ holds.

A complex Jordan algebra $\mathcal{B}$ with product $x \circ y$, unit element e and involution $x \mapsto x^*$ is called a JB^*-algebra if $\mathcal{B}$ carries a Banach space norm $\|\cdot\|$ satisfying $\|x \circ y\| \le \|x\| \cdot \|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$. Here $\{xy^*z\} := x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*)$ denotes the *Jordan triple product* of $x, y, z \in \mathcal{B}$. Throughout this section, let $\mathcal{B}$ and $\mathcal{C}$ be JB^*-algebras.

We are going to show the generalized Hyers-Ulam stability of algebra $*$-homomorphisms between JB^*-algebras.

THEOREM 4.1. *Let* $f : \mathcal{B} \to \mathcal{C}$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi : \mathcal{B}^4 \to [0, \infty)$ *satisfying* (i) *such that*

$$\|f(\mu x + \mu y + z \circ w) - \mu f(x) - \mu f(y) - f(z) \circ f(w)\| \le \varphi(x, y, z, w),$$

$$\text{(vi)} \qquad \|f(x^*) - f(x)^*\| \le \varphi(x, x, 0, 0)$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in \mathcal{B}$*. Then there exists a unique algebra* $*$*-homomorphism* $h : \mathcal{B} \to \mathcal{C}$ *such that satisfying the inequality* (iii)*.*

Proof. By the same method as in the proof of Theorem 2.1, one can show that there exists a unique algebra homomorphism $h : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iii).

It follows from (i) and (vi) that

$$\begin{aligned} h(x^*) &= \lim_{n\to\infty} \frac{f(2^n x^*)}{2^n} = \lim_{n\to\infty} \frac{f((2^n x)^*)}{2^n} = \lim_{n\to\infty} \frac{f(2^n x)^*}{2^n} = (\lim_{n\to\infty} \frac{f(2^n x)}{2^n})^* \\ &= h(x)^* \end{aligned}$$

for all $x \in \mathcal{B}$. Hence the algebra homomorphism $h : \mathcal{B} \to \mathcal{C}$ is involutive, as desired. □

Corollary 4.2. *Let* $f : \mathcal{B} \to \mathcal{C}$ *be a mapping with* $f(0) = 0$ *for which there exist constants* $\theta \geq 0$ *and* $p \in [0, 1)$ *such that*

$$\begin{aligned} &\|f(\mu x + \mu y + z \circ w) - \mu f(x) - \mu f(y) - f(z) \circ f(w)\| \\ &\qquad\qquad \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \\ &\|f(x^*) - f(x)^*\| \leq 2\theta \|x\|^p \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in \mathcal{B}$*. Then there exists a unique algebra* $*$*-homomorphism* $h : \mathcal{B} \to \mathcal{C}$ *such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all $x \in \mathcal{B}$*.*

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$, and apply Theorem 4.1. □

Theorem 4.3. *Let* $f : \mathcal{B} \to \mathcal{C}$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi : \mathcal{B}^4 \to [0, \infty)$ *satisfying* (i) *and* (vi) *such that*

$$\|f(\mu x + \mu y + z \circ w) - \mu f(x) - \mu f(y) - f(z) \circ f(w)\| \leq \varphi(x, y, z, w)$$

for $\mu = 1, i$*, and all* $x, y, z, w \in \mathcal{B}$*. If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{B}$*, then there exists a unique algebra* $*$*-homomorphism* $h : \mathcal{B} \to \mathcal{C}$ *satisfying the inequality* (iii)*.*

Proof. By the same reasoning as in the proofs of Theorems 2.1 and 4.1, there exists a unique algebra $*$-homomorphism $h : \mathcal{B} \to \mathcal{C}$ satisfying the inequality (iii), as desired. □

References

[1] P. Găvruta, *A generalization of the hyers-ulam-rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[2] Y. Jung and K. Park, *On the stability of the functional equation* $f(x+y+xy) = f(x) + f(y) + xf(y) + yf(x)$, J. Math. Anal. Appl. **274** (2002), 659–666.
[3] R.V. Kadison and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266.
[4] C. Park, *On the stability of the linear mapping in banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.
[5] Th.M. Rassias, *On the stability of the linear mapping in banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[6] H. Upmeier, *Jordan algebras in analysis, operator theory, and quantum mechanics*, Regional Conference Series in Mathematics No. **67**, Amer. Math. Soc., Providence, 1987.

Department of Mathematics
Daejin University
Kyeonggi 487-711, Republic of Korea
E-mail: `hyper@daejin.ac.kr`

Department of Mathematics
Hanyang University
Seoul 133-791, Republic of Korea
E-mail: `baak@hanyang.ac.kr`