INFINITE FINITE RANGE INEQUALITIES

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Abstract. Infinite finite range inequalities relate the norm of a weighted polynomial over $\mathbb{R}$ to its norm over a finite interval. In this paper we extend such inequalities to generalized polynomials with the weight $W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^\alpha)$.

1. Introduction

In the analysis of extremal polynomials, inequalities relating $L_p$ norms of weighted polynomials over infinite and finite intervals are important because they reduce problems over an infinite interval to problems on a finite interval. Freud, Nevai and others (see [11]) obtained inequalities that sufficed for weighted Bernstein type theorems on $\mathbb{R}$. Subsequently Mhaskar and Saff [8] established sharper inequalities that led to $n$th root asymptotics for $L_p$ extremal polynomials. In resolving Freud’s conjecture, Lubinsky, Mhaskar and Saff [7] further sharpened these inequalities. In this paper we extend such inequalities to generalized polynomials with the weight $W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^\alpha)$.

A generalized nonnegative algebraic polynomial is a function of the type

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j} \quad (0 \neq \omega \in \mathbb{C})$$

with $r_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$, and the number

$$n \overset{\text{def}}{=} \sum_{j=1}^{m} r_j$$
is called the generalized degree of \( f \). Note that \( n > 0 \) is not necessarily an integer.

We denote by \( \text{GANP}_n \) the set of all generalized nonnegative algebraic polynomials of degree at most \( n \in \mathbb{R}^+ \).

Using
\[
|z - z_j|^{r_j} = ((z - z_j)(z - \bar{z}_j))^{r_j/2}, \quad z \in \mathbb{R},
\]
we can easily check that when \( f \in \text{GANP}_n \) is restricted to the real line, then it can be written as
\[
f = \prod_{j=1}^{m} P_j^{r_j/2}, \quad 0 \leq P_j \in \mathbb{P}_2, \quad r_j \in \mathbb{R}^+, \quad \sum_{j=1}^{m} r_j \leq n,
\]
which is the product of nonnegative polynomials raised to positive real powers. This explains the name *generalized nonnegative polynomials*. Many properties of generalized nonnegative polynomials were investigated in a series of papers ([1,2,3,4]).

Associated with the Freud weight \( W_\alpha(x) = \exp(-|x|^{\alpha}) \), \( \alpha > 0 \), there are Mhaskar-Rahmanov-Saff numbers \( a_n = a_n(\alpha) \), which is the positive solution of the equation
\[
n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t)(1 - t^2)^{-\frac{1}{2}} dt, \quad n \in \mathbb{R}^+,
\]
where \( Q(x) = |x|^{\alpha} \), \( \alpha > 0 \). Explicitly,
\[
a_n = a_n(\alpha) = \left( \frac{n}{\lambda_\alpha} \right)^{1/\alpha}, \quad n \in \mathbb{R}^+, \quad 
\]
where
\[
\lambda_\alpha = \frac{2^{2-\alpha} \Gamma(\alpha)}{\{\Gamma(\alpha/2)\}^2}.
\]
Its importance lies partly in the identity [9]
\[
\|PW_\alpha\|_{L^\infty(\mathbb{R})} = \|PW_\alpha\|_{L^\infty([-a_n, a_n])}, \quad P \in \mathbb{P}_n.
\]
Now we state our results.

**Theorem 1.1.** Let \( \epsilon > 0 \), \( d > 0 \), and \( 0 < p < \infty \). Let
\[
W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^{\alpha}),
\]
where $\alpha > 1$, $x_k \in \mathbb{R}$, and $p\gamma_k > -1$, for $k = 1, \cdots, m$. Let 
\[ s_n = \min \left\{ \frac{da_n}{n}, a_n \right\}, \epsilon \leq n \in \mathbb{R}^+. \]

Then there exist positive constants $B$ and $C$ such that 
\[ \int_{-\infty}^{\infty} f^p(x)W^p(x)dx \leq C \int_{I_n \setminus \Delta_n} f^p(x)W^p(x)dx, \]
for all $f \in \text{GANP}_n$, $\epsilon \leq n \in \mathbb{R}^+$, where 
\[ I_n = [-Ba_n, Ba_n] \]
and $\Delta_n$ is any measurable subset of $I_n$ with $m(\Delta_n) \leq s_n$.

As a consequence of Theorem 1.1, we have the following.

**Corollary 1.2.** Let $\epsilon > 0$ and $0 < p < \infty$. Let 
\[ W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^\alpha), \]
where $\alpha > 1$, $x_k \in \mathbb{R}$, and $p\gamma_k > -1$, for $k = 1, \cdots, m$. Then there exist positive constants $B$ and $C$ such that 
\[ \int_{-\infty}^{\infty} f^p(x)W^p(x)dx \leq C \int_{-Ba_n}^{Ba_n} f^p(x)W^p(x)dx, \]
for all $f \in \text{GANP}_n$, $\epsilon \leq n \in \mathbb{R}^+$.

We can drop the condition $p\gamma_k > -1$ in Theorem 1.1 if we replace $W$ by $W_n$ as follows.

**Theorem 1.3.** Let $\epsilon > 0$, $d > 0$, and $0 < p \leq \infty$. Let 
\[ W_n(x) = \prod_{k=1}^{m} \left( |x - x_k| + \frac{a_n}{n} \right)^{\gamma_k} \cdot \exp(-|x|^\alpha), \]
where $n \in \mathbb{R}^+$, $\alpha > 1$, and $x_k, \gamma_k \in \mathbb{R}$, for $k = 1, \cdots, m$. Let 
\[ s_n = \min \left\{ \frac{da_n}{n}, a_n \right\}, \epsilon \leq n \in \mathbb{R}^+. \]

Then there exist positive constants $B$ and $C$ such that 
\[ \|fW_n\|_{L^p(\mathbb{R})} \leq C \|fW_n\|_{L^p(I_n \setminus \Delta_n)}, \]
for all $f \in \text{GANP}_n$, $\epsilon \leq n \in \mathbb{R}^+$, where 
\[ I_n = [-Ba_n, Ba_n] \]
and $\Delta_n$ is any measurable subset of $I_n$ with $m(\Delta_n) \leq s_n$.

Throughout this paper we write $g_n(x) \sim h_n(x)$ if for every $n$ and for every $x$ in consideration

$$0 < c_1 \leq \frac{g_n(x)}{h_n(x)} \leq c_2 < \infty,$$

and $g(x) \sim h(x), n \sim N$ have similar meanings.

2. Proof of theorems

In order to prove Theorems, first we need infinite finite range inequalities for generalized polynomials with the Freud weight $W_\alpha(x) = \exp(-|x|^\alpha)$. We restate Theorem 2.2 in [5. p. 124].

**Lemma 2.1.** Let $\epsilon > 0$ and $d > 0$. Let $W_\alpha(x) = \exp(-|x|^\alpha), \alpha > 1$. Let

$$s_n = \min \left\{ \frac{da_n}{n}, a_n \right\}, \quad n \in \mathbb{R}^+.$$

If $0 < p < \infty$, then there exist positive constants $B^*$ and $C_1$ such that for all measurable sets $\Delta_n \subset [-B^*a_n, B^*a_n]$ with $m(\Delta_n) \leq s_n/2$,

$$\int_{-\infty}^{\infty} f^p(x)W^p_\alpha(x)dx \leq C_1 \int \lfloor x \rfloor \leq B^*a_n \int f^p(x)W^p_\alpha(x)dx,$$

for all $f \in \text{GANP}_n, x \notin \Delta_n$.

If $p = \infty$, then there exists a positive constant $C_2$ such that for all measurable sets $\Delta_n \subset [-B^*a_n, B^*a_n]$ with $m(\Delta_n) \leq s_n$,

$$\|fW_\alpha\|_{L^\infty(\mathbb{R})} \leq C_2 \|fW_\alpha\|_{L^\infty([-B^*a_n, B^*a_n] \setminus \Delta_n)},$$

for all $f \in \text{GANP}_n, n \in \mathbb{R}^+$.

**Proof.** See the proof of Theorem 2.2 in [5. p. 124].

Next we define generalized Christoffel functions. Let $0 < p < \infty$. Then the generalized Christoffel function for ordinary polynomials is defined by

$$\lambda_{n,p}(W_\alpha; x) = \min_{P \in \mathbb{P}_{n-1}} \int_{-\infty}^{\infty} \frac{|P(t)W_\alpha(t)|^p}{|P(x)|^p} dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$
The generalized Christoffel function for generalized nonnegative polynomials is defined by

$$\omega_{n,p}(W_\alpha;x) = \inf_{f \in \text{GANP}_n} \int_{-\infty}^{\infty} \frac{(f(t)W_\alpha(t))^p}{f^p(x)} dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{R}^+. $$

For the estimates of $\omega_{n,p}(W_\alpha;x)$, we need the following lemma, which is the restatement of Theorem 2.3 in [5, p. 125].

**Lemma 2.2.** Let $W_\alpha(x) = \exp(-|x|^{\alpha})$, $\alpha > 1$. Let $0 < p < \infty$. Then

$$\omega_{n,p}(W_\alpha;x) \geq C \frac{a_n}{n} W_\alpha^p(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{R}^+, $$

and

$$\omega_{n,p}(W_\alpha;x) \leq \lambda_{[n]+1,p}(W_\alpha;x), \quad x \in \mathbb{R}, \quad n \in \mathbb{R}^+, $$

where $[n]$ denotes the integer part of $n$.

**Proof.** See the proof of Theorem 2.3 in [5, p. 125].

**Remark.** It is well known (see, for example, [6]) that if $\alpha > 1$, then there exist positive constants $C_1$ and $C_2$ depending on $p$ and $\alpha$, such that

$$\lambda_{[n]+1,p}(W_\alpha;x) \leq C_1 \frac{a_n}{n} W_\alpha^p(x), \quad |x| \leq C_2 a_n. $$

Consequently

$$\omega_{n,p}(W_\alpha;x) \sim \frac{a_n}{n} W_\alpha^p(x), \quad |x| \leq C_2 a_n. $$

Now we prove our results.

**Proof of Theorem 1.1.** Let $\epsilon > 0$, $d > 0$, and $0 < p < \infty$. Let

$$W(x) = \prod_{k=1}^{m} v_k(x) \cdot \exp(-|x|^\alpha) \quad (\alpha > 1), $$

where

$$v_k(x) = |x - x_k|^{\gamma_k}, $$

and

$$\gamma_k < 0, \quad \text{for} \quad 1 \leq k \leq i, $$

$$0 \leq \gamma_k < 1, \quad \text{for} \quad i < k \leq j, $$

$$1 \leq \gamma_k, \quad \text{for} \quad j < k \leq m. $$
Suppose that $p_{\gamma_k} > -1$, $k = 1, 2, \ldots, m$. Let

$$ (2.3) \Gamma_n = n + 4ni + \sum_{k=i+1}^{m} \gamma_k $$

Let $B^*$ be the constant which satisfies (2.1). Choose $B > 0$ big enough so that

$$ (2.4) B^* a_{\Gamma_n} \leq B a_n, \quad \text{for } n \geq \epsilon, $$

and

$$ (2.5) |x_k| < (B a_n)/2, \quad \text{for } k = 1, 2, \ldots, m, \text{ and } n \geq \epsilon. $$

Let

$$ I_n = [-B a_n, B a_n], $$

and let $\Delta_n$ be any measurable subset of $I_n$ with $m(\Delta_n) \leq s_n$, where

$$ s_n = \min \left\{ \frac{d a_n}{n}, a_n \right\}. $$

Let $d_1 > 0$ and

$$ A_{n,k} = \left( x_k - \frac{d_1 a_n}{n}, x_k + \frac{d_1 a_n}{n} \right), \quad k = 1, 2, \ldots, m, \quad n \geq \epsilon, $$

and

$$ J_n = \bigcup_{k=1}^{m} A_{n,k}. $$

Here, we can find $d_1 > 0$ so that $A_{n,k}$'s are self disjoint for $k = 1, 2, \ldots, m$, and $J_n \subset I_n$ and

$$ (2.6) m(\Delta_n \cup J_n) \leq \min \left\{ \frac{(d + 1)a_n}{n}, 2a_n \right\}. $$

Now denote by $P_{j}(\alpha, \beta, x)$, $(\alpha > -1, \beta > -1)$, $j = 0, 1, 2, \ldots$, the orthonormalized Jacobi polynomials and let

$$ K_M(\alpha, \beta, x) = \sum_{j=0}^{M-1} P_{j}^2(\alpha, \beta, x). $$

Let

$$ Q_{M,k}(x) = \frac{1}{M} K_M \left( -\frac{1}{2}, \frac{\gamma_k - 1}{2}, 2x^2 - 1 \right), \quad M \in \mathbb{N}, \quad 1 \leq k \leq j. $$

It is well known (see [10, Lemma 2, p. 241] and [12, p.108]) that

$$ |Q'_{M,k}(x)| \leq c_1 |x|^{-1} |1 - x^2|^{-1} Q_{M,k}(x), \quad \text{for } |x| \leq 1, $$

where $c_1$ is a constant.
and
\[ Q_{M,k}(x) \sim \left( |x| + \frac{1}{M} \right)^{\gamma_k}, \quad \text{for } |x| \leq 1. \]

Now for each \( \epsilon \leq n \in \mathbb{R}^+ \), let \( N = [n] + 1 \) and
\[ R_{n,k}(x) = (4Ba_n)^{\gamma_k}Q_{N,k} \left( \frac{x - x_k}{4Ba_n} \right), \quad \text{for } k = 1, 2, \cdots, j. \]

Then we have
\[ |R'_{n,k}(x)| \leq c_2 \frac{n}{a_n}R_{n,k}(x), \quad \text{for } x \in I_n \setminus A_{n,k}, \]
\[ R_{n,k}(x) \sim \left( |x - x_k| + \frac{a_n}{n} \right)^{\gamma_k}, \quad \text{for } x \in I_n, \]
and
\[ R_{n,k}(x) \sim v_k(x), \quad \text{for } x \in I_n \setminus A_{n,k}. \]

Now let
\[ D_n = \Delta_n \setminus J_n \text{ and } B_{n,k} = A_{n,k} \cap \Delta_n. \]

Let \( f \in \text{GANP}_n, n \geq \epsilon \). First we show that
\[ \int_{D_n} (fW)^p(x)dx \leq c_3 \int_{I_n \setminus \Delta_n} (fW)^p(x)dx. \]

Since
\[ R_{n,k}(x) \sim v_k(x), \quad x \in D_n, \quad 1 \leq k \leq i, \]
we have
\[ \int_{D_n} (fW)^p(x)dx \leq c_4 \int_{D_n} (fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(x)dx. \]

Since \((fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_m)\) is a generalized polynomial of degree less than \( \Gamma_n = O(n) \), by Lemma 2.1, (2.4), and (2.6), we obtain
\[ \int_{D_n} (fW)^p(x)dx \]
\[ \leq c_5 \int_{I_n \setminus (\Delta_n \cup J_n)} (fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(x)dx \]
\[ \leq c_6 \int_{I_n \setminus (\Delta_n \cup J_n)} (fW)^p(x)dx \]
\[ \leq c_6 \int_{I_n \setminus \Delta_n} (fW)^p(x). \]
Next we show that

\[
(2.12) \quad \int_{B_{n,k}} (fW)^p(x)dx \leq c_7 \int_{I_n \setminus \Delta_n} (fW)^p(x)dx, \quad 1 \leq k \leq m.
\]

We distinguish two cases.

**Case 1.** \(1 \leq k \leq i, (\gamma_k < 0)\). Since \((fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_m)^\alpha\) is a generalized polynomial of degree less than \(\Gamma_n = O(n)\), by Lemma 2.2, we have

\[
(2.13) \quad (fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(x) \leq c_8 a_n \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(t)dt, \quad \text{for } x \in \mathbb{R}.
\]

Multiplying by \(v_k^p(x)\) and then integrating both sides over \(x \in A_{n,k}\), we obtain

\[
\int_{x \in A_{n,k}} (v_k fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(x)dx
\leq c_9 \left( \frac{a_n}{n} \right)^{\gamma_k} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(x)dx.
\]

Since

\[
\left( \frac{n}{a_n} \right)^{\gamma_k} R_{n,k}(x) \geq c_{10}, \quad \text{for } x \in A_{n,k}, \quad \text{by (2.9)},
\]

and

\[
R_{n,\ell}(x) \sim v_\ell(x), 1 \leq \ell \leq i, \ell \neq k, \quad \text{for } x \in A_{n,k}, \quad \text{by (2.10)},
\]

we have

\[
\int_{x \in A_{n,k}} (fW)^p(x)dx
\leq c_{11} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(x)dx.
\]
Then by Lemma 2.1, (2.4), and (2.6),

\[
\int_{x \in A_{n,k}} (fW)^p(x) \, dx \\
\leq c_{11} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x) \, dx \\
\leq c_{12} \int_{I_n \setminus (\Delta_n \cup J_n)} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x) \, dx \\
\leq c_{13} \int_{I_n \setminus (\Delta_n \cup J_n)} (fW)^p(x) \, dx.
\]

Noting that

\[B_{n,k} \subset A_{n,k} \text{ and } I_n \setminus (\Delta_n \cup J_n) \subset I_n \setminus \Delta_n,\]

we have

\[(2.14) \quad \int_{x \in B_{n,k}} (fW)^p(x) \, dx \leq c_{13} \int_{I_n \setminus \Delta_n} (fW)^p(x) \, dx, \quad 1 \leq k \leq i.\]

Case 2. \(i < k \leq m, (\gamma_k \geq 0)\). Integrating both sides of (2.13) over \(x \in A_{n,k}\), we obtain

\[
\int_{x \in A_{n,k}} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x) \, dx \\
\leq c_{14} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x) \, dx.
\]

Since

\[R_{n,\ell}(x) \sim v_{\ell}(x), 1 \leq \ell \leq i, \quad \text{for } x \in A_{n,k}, \quad \text{by (2.10),}\]

we have

\[
\int_{x \in A_{n,k}} (fW)^p(x) \, dx \\
\leq c_{15} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x) \, dx.
\]
Then by Lemma 2.1, (2.4), and (2.5),
\[
\int_{x \in A_{n,k}} (fW)^p(x)dx \\
\leq c_{15} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(x)dx \\
\leq c_{16} \int_{I_n \setminus (\Delta_n \cup J_n)} (fR_{n,1} \cdots R_{n,i}v_{i+1} \cdots v_mW_\alpha)^p(x)dx \\
\leq c_{17} \int_{I_n \setminus (\Delta_n \cup J_n)} (fW)^p(x)dx,
\]
hence,
\[
\int_{x \in A_{n,k}} (fW)^p(x)dx \leq c_{17} \int_{I_n \setminus \Delta_n} (fW)^p(x)dx.
\]
Combining (2.11) and (2.12) yields
(2.15) \[ \int_{\Delta_n} (fW)^p(x)dx \leq c_{18} \int_{I_n \setminus \Delta_n} (fW)^p(x)dx. \]
Next we show
(2.16) \[ \int_{|x| \geq B_a_n} (fW)^p(x)dx \leq c_{19} \int_{I_n \setminus \Delta_n} (fW)^p(x)dx. \]
Let, for \(1 \leq k \leq i\),
\[
M_{n,k} = \max(|B_a_n + x_k|^{\gamma_k}, |B_a_n - x_k|^{\gamma_k})
\]
and
\[
m_{n,k} = \min(|B_a_n + x_k|^{\gamma_k}, |B_a_n - x_k|^{\gamma_k}).
\]
Then
\[
\frac{M_{n,k}}{m_{n,k}} \leq C(k), \quad \text{by (2.5),}
\]
hence,
\[
\int_{|x| \geq B_a_n} (fW)^p(x)dx \\
\leq M_{n,1} \cdots M_{n,i} \int_{|x| \geq B_a_n} (fv_{i+1} \cdots v_mW_\alpha)^p(x)dx \\
\leq M_{n,1} \cdots M_{n,i} \int_{-\infty}^{\infty} (fv_{i+1} \cdots v_mW_\alpha)^p(x)dx.
\]
therefore, by Lemma 2.1,
\[
\int_{|x| \geq B_{a_n}} (fW)^p(x)dx \\
\leq c_20 M_{n,1} \cdots M_{n,i} \int_{\Delta_n \setminus I_n} (f v_{i+1} \cdots v_m W_\alpha p(x)dx \\\n\leq c_20 \frac{M_{n,1} \cdots M_{n,i}}{m_{n,1} \cdots m_{n,i}} \int_{\Delta_n \setminus I_n} (fW)^p(x)dx \\\n\leq c_21 \int_{\Delta_n \setminus I_n} (fW)^p(x)dx.
\]
Then by (2.15) and (2.16), we have
\[
\int_{-\infty}^{\infty} (fW)^p(x)dx = \int_{\Delta_n} (fW)^p(x)dx + \int_{\Delta_n \setminus I_n} (fW)^p(x)dx + \int_{|x| \geq B_{a_n}} (fW)^p(x)dx \leq c_22 \int_{\Delta_n \setminus I_n} (fW)^p(x)dx,
\]
hence, Theorem 1.1 is proved.

Proof of Corollary 1.2. Corollary 1.2 follows directly from Theorem 1.1.

Proof of Theorem 1.3. Let \( \epsilon > 0, d > 0, \) and \( 0 < p \leq \infty. \) For simplicity we consider
\[
W_n(x) = \prod_{k=1}^{2} \left( |x - x_k| + \frac{a_n}{n} \right)^{\gamma_k} \cdot \exp(-|x|^\alpha),
\]
where \( n \in \mathbb{R}^+, \alpha > 1, \) and
\( \gamma_1 < 0 \) and \( \gamma_2 \geq 0. \)
General case follows by the same method. Let
\( \beta(n) = 5n + \gamma_2. \)
Let \( B^* \) be the constant which satisfies (2.1). Choose \( B > 0 \) big enough so that
\[
(2.17) \quad B^* a_{\beta_n} \leq B a_n, \quad \text{for } n \geq \epsilon,
\]
and
(2.18) \(|x_k| < (Ba_n)/2, \text{ for } k = 1, 2, \text{ and } n \geq \epsilon.\)

Let
\[ I_n = [-Ba_n, Ba_n], \]
and let \( \Delta_n \) be any measurable subset of \( I_n \) with \( m(\Delta_n) \leq s_n, \) where
\[ s_n = \min\left\{ \frac{da_n}{n}, a_n \right\}. \]

Let
\[ v_{n,k}(x) = \left( |x - x_k| + \frac{a_n}{n} \right)^{\gamma_k}, \quad k = 1, 2. \]

And let
\[ u_{n,1}(x) = |x - x_2 + \frac{a_n}{n}|^{\gamma_2} \]
and
\[ u_{n,2}(x) = |x - x_2 - \frac{a_n}{n}|^{\gamma_2}. \]

We use the polynomial \( R_{n,1} \) which we constructed in the proof of Theorem 1.1. See (2.7) and (2.9). Recall that \( R_{n,1} \) has degree at most \( 4n \) and
(2.19) \( R_{n,1}(x) \sim v_{n,1}(x), \quad x \in I_n. \)

Note that
\[
\begin{align*}
\frac{1}{2} \left( |x - x_2 + \frac{a_n}{n}| + |x - x_2 - \frac{a_n}{n}| \right) & \leq \left( |x - x_2| + \frac{a_n}{n} \right) \\
& \leq |x - x_2 + \frac{a_n}{n}| + |x - x_2 - \frac{a_n}{n}|, \quad x \in \mathbb{R}.
\end{align*}
\]

Then using
\[
c_1(p)(|a| + |b|)^p \leq (|a|^p + |b|^p) \leq c_2(p)(|a| + |b|)^p, \quad (0 < p < \infty),
\]
we have
(2.20) \( v_{n,2}(x) \sim (u_{n,1}(x) + u_{n,2}(x)), \quad x \in \mathbb{R}. \)

Let \( f \in \text{GANP}_n. \) Since \( (fR_{n,1}u_{n,1}) \) has degree at most \( \beta(n) = O(n), \) by Lemma 2.1 and (2.17), we have
\[
\|fR_{n,1}u_{n,1}W_{\alpha}\|_{L^p(\Delta_n)} \leq c_1\|fR_{n,1}u_{n,1}W_{\alpha}\|_{L^p(I_n \setminus \Delta_n)}.
\]

Since
\[ R_{n,1}(x) \sim v_{n,1}(x), \quad x \in I_n, \]
\begin{align*}
\text{Infinite finite range inequalities} & \quad 75 \\
\text{and} & \\
\quad u_{n,1}(x) \leq v_{n,2}(x), \quad x \in \mathbb{R},
\end{align*}

we have

\[ ||f R_{n,1} W_\alpha||_{L^p(\Delta_n)} \leq c_2 ||f W_\alpha||_{L^p(I_n \setminus \Delta_n)}. \]

Similarly we obtain

\[ ||f R_{n,1} u_{n,2} W_\alpha||_{L^p(\Delta_n)} \leq c_2 ||f W_\alpha||_{L^p(I_n \setminus \Delta_n)}. \]

Then by (2.19) and (2.20),

\[ ||f W_n||_{L^p(\Delta_n)} = ||f v_{n,1} v_{n,2} W_\alpha||_{L^p(\Delta_n)} \leq c_3 ||f R_{n,1} (u_{n,1} + u_{n,2}) W_\alpha||_{L^p(\Delta_n)} \leq c_4(||f R_{n,1} u_{n,1} W_\alpha||_{L^p(\Delta_n)} + ||f R_{n,1} u_{n,2} W_\alpha||_{L^p(\Delta_n)}) \]

\[ \leq c_5 ||f W_n||_{L^p(I_n \setminus \Delta_n)}. \tag{2.21} \]

Next we show that

\[ ||f W_n||_{L^p(\mathbb{R} \setminus I_n)} \leq c_6 ||f W_n||_{L^p(I_n \setminus \Delta_n)}. \]

Let

\[ M_n = \max_{|x| \geq B_n} \left\{ \left( |x - x_1| + \frac{a_n}{n} \right)^{\gamma_1} \right\} \]

and

\[ m_n = \min_{|x| \leq B_n} \left\{ \left( |x - x_1| + \frac{a_n}{n} \right)^{\gamma_1} \right\}. \]

Then by (2.18)

\[ \frac{M_n}{m_n} \leq c_7, \]

hence, by Lemma 2.1 we have

\[ ||f v_{n,1} u_{n,1} W_\alpha||_{L^p(\mathbb{R} \setminus I_n)} \leq M_n ||f u_{n,1} W_\alpha||_{L^p(\mathbb{R} \setminus I_n)} \leq M_n ||f u_{n,1} W_\alpha||_{L^p(\mathbb{R})} \leq c_8 M_n ||f u_{n,1} W_\alpha||_{L^p(I_n \setminus \Delta_n)} \leq c_8 \frac{M_n}{m_n} ||f v_{n,1} u_{n,1} W_\alpha||_{L^p(I_n \setminus \Delta_n)} \leq c_9 ||f v_{n,1} u_{n,1} W_\alpha||_{L^p(I_n \setminus \Delta_n)}. \]

Since

\[ u_{n,1}(x) \leq v_{n,2}(x), \quad x \in \mathbb{R}, \]

we obtain

\[ ||f v_{n,1} u_{n,1} W_\alpha||_{L^p(\mathbb{R} \setminus I_n)} \leq c_9 ||f W_n||_{L^p(I_n \setminus \Delta_n)}. \]
Similarly we have

\[ \| f v_{n,1} u_{n,2} W_\alpha \|_{L^p(\mathbb{R}\setminus I_n)} \leq c_9 \| f W_n \|_{L^p(I_n \setminus \Delta_n)}. \]

Then by (2.20),

\[ \| f W_n \|_{L^p(\mathbb{R}\setminus I_n)} = \| f v_{n,1} u_{n,2} W_\alpha \|_{L^p(\mathbb{R}\setminus I_n)} \]

\[ \leq c_{10} \| f v_{n,1} (u_{n,1} + u_{n,2}) W_\alpha \|_{L^p(\mathbb{R}\setminus I_n)} \]

\[ \leq c_{11} (\| f v_{n,1} u_{n,1} W_\alpha \|_{L^p(\mathbb{R}\setminus I_n)} + \| f v_{n,1} u_{n,2} W_\alpha \|_{L^p(\mathbb{R}\setminus I_n)}) \]

\[ \leq c_{12} \| f W_n \|_{L^p(I_n \setminus \Delta_n)}. \]

Combining (2.21) and the above inequality gives Theorem 1.3.

References

[6] A.L. Levin and D.S. Lubinsky, Canonical products and the weights \( \exp(-|x|^\alpha) \), \( \alpha > 1 \), with applications, J. Approx. Theory 49 (1987), 149-169.
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