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INFINITE FINITE RANGE INEQUALITIES

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Abstract. Infinite finite range inequalities relate the norm of a weighted polynomial over $\mathbb{R}$ to its norm over a finite interval. In this paper we extend such inequalities to generalized polynomials with the weight $W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^{\alpha})$.

1. Introduction

In the analysis of extremal polynomials, inequalities relating L_p norms of weighted polynomials over infinite and finite intervals are important because they reduce problems over an infinite interval to problems on a finite interval. Freud, Nevai and others (see [11]) obtained inequalities that sufficed for weighted Bernstein type theorems on $\mathbb{R}$. Subsequently Mhaskar and Saff [8] established sharper inequalities that led to nth root asymptotics for L_p extremal polynomials. In resolving Freud's conjecture, Lubinsky, Mhaskar and Saff [7] further sharpened these inequalities. In this paper we extend such inequalities to generalized polynomials with the weight $W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^{\alpha})$.

A generalized nonnegative algebraic polynomial is a function of the type

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j} \quad (0 \neq \omega \in \mathbb{C})$$

with $r_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$, and the number

$$n \stackrel{\text{def}}{=} \sum_{j=1}^{m} r_j$$

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is called the generalized degree of f. Note that $n > 0$ is not necessarily an integer.

We denote by GANP_n the set of all generalized nonnegative algebraic polynomials of degree at most $n \in \mathbb{R}^+$.

Using

$$|z - z_j|^{r_j} = ((z - z_j)(z - \bar{z}_j))^{r_j/2}, \quad z \in \mathbb{R},$$

we can easily check that when $f \in \mathrm{GANP}_n$ is restricted to the real line, then it can be written as

$$f = \prod_{j=1}^{m} P_j^{r_j/2}, \quad 0 \le P_j \in \mathbb{P}_2, \quad r_j \in \mathbb{R}^+, \quad \sum_{j=1}^{m} r_j \le n,$$

which is the product of nonnegative polynomials raised to positive real powers. This explains the name *generalized nonnegative polynomials*. Many properties of generalized nonnegative polynomials were investigated in a series of papers ([1,2,3,4]).

Associated with the Freud weight $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, there are Mhaskar-Rahmanov-Saff numbers $a_n = a_n(\alpha)$, which is the positive solution of the equation

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t)(1 - t^2)^{-\frac{1}{2}} dt, \quad n \in \mathbb{R}^+,$$

where $Q(x) = |x|^\alpha$, $\alpha > 0$. Explicitly,

$$a_n = a_n(\alpha) = \left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha}, \quad n \in \mathbb{R}^+,$$

where

$$\lambda_\alpha = \frac{2^{2-\alpha}\Gamma(\alpha)}{\{\Gamma(\alpha/2)\}^2}.$$

Its importance lies partly in the identity [9]

$$\|PW_\alpha\|_{L^\infty(\mathbb{R})} = \|PW_\alpha\|_{L^\infty([-a_n, a_n])}, \quad P \in \mathbb{P}_n.$$

Now we state our results.

THEOREM 1.1. *Let* $\epsilon > 0$, $d > 0$, *and* $0 < p < \infty$. *Let*

$$W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^\alpha),$$

where $\alpha > 1$, $x_k \in \mathbb{R}$, *and* $p\gamma_k > -1$, *for* $k = 1, \cdots, m$. *Let*

$$s_n = \min\left\{\frac{da_n}{n}, a_n\right\}, \epsilon \leq n \in \mathbb{R}^+.$$

Then there exist positive constants B *and* C *such that*

$$\int_{-\infty}^{\infty} f^p(x)W^p(x)dx \leq C \int_{I_n \setminus \Delta_n} f^p(x)W^p(x)dx,$$

for all $f \in \mathrm{GANP}_n$, $\epsilon \leq n \in \mathbb{R}^+$, *where*

$$I_n = [-Ba_n, Ba_n]$$

and Δ_n *is any measurable subset of* I_n *with* $m(\Delta_n) \leq s_n$.

As a consequence of Theorem 1.1, we have the following.

Corollary 1.2. *Let* $\epsilon > 0$ *and* $0 < p < \infty$. *Let*

$$W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^\alpha),$$

where $\alpha > 1$, $x_k \in \mathbb{R}$, *and* $p\gamma_k > -1$, *for* $k = 1, \cdots, m$. *Then there exist positive constants* B *and* C *such that*

$$\int_{-\infty}^{\infty} f^p(x)W^p(x)dx \leq C \int_{-Ba_n}^{Ba_n} f^p(x)W^p(x)dx,$$

for all $f \in \mathrm{GANP}_n$, $\epsilon \leq n \in \mathbb{R}^+$.

We can drop the condition $p\gamma_k > -1$ in Theorem 1.1 if we replace W by W_n as follows.

Theorem 1.3. *Let* $\epsilon > 0$, $d > 0$, *and* $0 < p \leq \infty$. *Let*

$$W_n(x) = \prod_{k=1}^{m} \left(|x - x_k| + \frac{a_n}{n}\right)^{\gamma_k} \cdot \exp(-|x|^\alpha),$$

where $n \in \mathbb{R}^+$, $\alpha > 1$, *and* $x_k, \gamma_k \in \mathbb{R}$, *for* $k = 1, \cdots, m$. *Let*

$$s_n = \min\left\{\frac{da_n}{n}, a_n\right\}, \epsilon \leq n \in \mathbb{R}^+.$$

Then there exist positive constants B *and* C *such that*

$$\|fW_n\|_{L^p(\mathbb{R})} \leq C\|fW_n\|_{L^p(I_n \setminus \Delta_n)},$$

for all $f \in \mathrm{GANP}_n$, $\epsilon \leq n \in \mathbb{R}^+$, *where*

$$I_n = [-Ba_n, Ba_n]$$

and Δ_n *is any measurable subset of* I_n *with* $m(\Delta_n) \le s_n$.

Throughout this paper we write $g_n(x) \sim h_n(x)$ if for every n and for every x in consideration

$$0 < c_1 \le \frac{g_n(x)}{h_n(x)} \le c_2 < \infty,$$

and $g(x) \sim h(x)$, $n \sim N$ have similar meanings.

2. Proof of theorems

In order to prove Theorems, first we need infinite finite range inequalities for generalized polynomials with the Freud weight $W_\alpha(x) = \exp(-|x|^\alpha)$. We restate Theorem 2.2 in [5. p. 124].

LEMMA 2.1. *Let* $\epsilon > 0$ *and* $d > 0$. *Let* $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. *Let*

$$s_n = \min\left\{\frac{da_n}{n}, a_n\right\}, \quad n \in \mathbb{R}^+.$$

If $0 < p < \infty$, *then there exist positive constants* B^* *and* C_1 *such that for all measurable sets* $\Delta_n \subset [-B^* a_n, B^* a_n]$ *with* $m(\Delta_n) \le s_n/2$,

$$\int_{-\infty}^{\infty} f^p(x) W_\alpha^p(x)dx \le C_1 \int_{\substack{|x| \le B^* a_n \\ x \notin \Delta_n}} f^p(x) W_\alpha^p(x)dx, \tag{2.1}$$

for all $f \in \mathrm{GANP}_n$, $\epsilon \le n \in \mathbb{R}^+$.

If $p = \infty$, *then there exists a positive constant* C_2 *such that for all measurable sets* $\Delta_n \subset [-B^* a_n, B^* a_n]$ *with* $m(\Delta_n) \le s_n$,

$$\|fW_\alpha\|_{L^\infty(\mathbb{R})} \le C_2 \|fW_\alpha\|_{L^\infty([-B^* a_n, B^* a_n]\setminus \Delta_n)}, \tag{2.2}$$

for all $f \in \mathrm{GANP}_n$, $n \in \mathbb{R}^+$.

Proof. See the proof of Theorem 2.2 in [5. p. 124]. □

Next we define generalized Christoffel functions. Let $0 < p < \infty$. Then the generalized Christoffel function for ordinary polynomials is defined by

$$\lambda_{n,p}(W_\alpha; x) = \min_{P \in \mathbb{P}_{n-1}} \int_{-\infty}^{\infty} \frac{|P(t)W_\alpha(t)|^p}{|P(x)|^p} dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

The generalized Christoffel function for generalized nonnegative polynomials is defined by

$$\omega_{n,p}(W_\alpha;x) = \inf_{f\in \mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{(f(t)W_\alpha(t))^p}{f^p(x)}dt, \quad x\in\mathbb{R}, \quad n\in\mathbb{R}^+.$$

For the estimates of $\omega_{n,p}(W_\alpha;x)$, we need the following lemma, which is the restatement of Theorem 2.3 in [5, p. 125].

LEMMA 2.2. *Let* $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. *Let* $0 < p < \infty$. *Then*

$$\omega_{n,p}(W_\alpha;x) \geq C\frac{a_n}{n}W_\alpha^p(x), \quad x\in\mathbb{R}, \quad n\in\mathbb{R}^+,$$

and

$$\omega_{n,p}(W_\alpha;x) \leq \lambda_{[n]+1,p}(W_\alpha;x), \quad x\in\mathbb{R}, \quad n\in\mathbb{R}^+,$$

where $[n]$ *denotes the integer part of* n.

Proof. See the proof of Theorem 2.3 in [5, p. 125]. □

Remark. It is well known (see, for example, [6]) that if $\alpha > 1$, then there exist positive constants C_1 and C_2 depending on p and α, such that

$$\lambda_{[n]+1,p}(W_\alpha;x) \leq C_1\frac{a_n}{n}W_\alpha^p(x), \quad |x| \leq C_2 a_n.$$

Consequently

$$\omega_{n,p}(W_\alpha;x) \sim \frac{a_n}{n}W_\alpha^p(x), \quad |x| \leq C_2 a_n.$$

Now we prove our results.

Proof of Theorem 1.1. Let $\epsilon > 0$, $d > 0$, and $0 < p < \infty$. Let

$$W(x) = \prod_{k=1}^{m} v_k(x)\cdot\exp(-|x|^\alpha) \quad (\alpha > 1),$$

where

$$v_k(x) = |x - x_k|^{\gamma_k},$$

and

$$\begin{aligned} \gamma_k < 0, &\quad \text{for} \quad 1 \leq k \leq i, \\ 0 \leq \gamma_k < 1, &\quad \text{for} \quad i < k \leq j, \\ 1 \leq \gamma_k, &\quad \text{for} \quad j < k \leq m. \end{aligned}$$

Suppose that $p\gamma_k > -1$, $k = 1, 2, \cdots, m$. Let

$$\Gamma_n = n + 4ni + \sum_{k=i+1}^{m} \gamma_k \tag{2.3}$$

Let B^* be the constant which satisfies (2.1). Choose $B > 0$ big enough so that

$$B^* a_{\Gamma_n} \le B a_n, \quad \text{for } n \ge \epsilon, \tag{2.4}$$

and

$$|x_k| < (Ba_n)/2, \quad \text{for } k = 1, 2, \cdots, m, \text{ and } n \ge \epsilon. \tag{2.5}$$

Let

$$I_n = [-Ba_n, Ba_n],$$

and let Δ_n be any measurable subset of I_n with $m(\Delta_n) \le s_n$, where

$$s_n = \min\left\{\frac{da_n}{n}, a_n\right\}.$$

Let $d_1 > 0$ and

$$A_{n,k} = \left(x_k - \frac{d_1 a_n}{n}, x_k + \frac{d_1 a_n}{n}\right), \quad k = 1, 2, \cdots, m, \quad n \ge \epsilon,$$

and

$$J_n = \cup_{k=1}^{m} A_{n,k}.$$

Here, we can find $d_1 > 0$ so that $A_{n,k}$'s are self disjoint for $k = 1, 2, \cdots, m$, and $J_n \subset I_n$ and

$$m(\Delta_n \cup J_n) \le \min\left\{\frac{(d+1)a_n}{n}, 2a_n\right\}. \tag{2.6}$$

Now denote by $P_j(\alpha, \beta, x)$, $(\alpha > -1, \beta > -1)$, $j = 0, 1, 2, \cdots$, the orthonormalized Jacobi polynomials and let

$$K_M(\alpha, \beta, x) = \sum_{j=0}^{M-1} P_j^2(\alpha, \beta, x).$$

Let

$$Q_{M,k}(x) = \frac{1}{M} K_M\left(-\frac{1}{2}, \frac{\gamma_k - 1}{2}, 2x^2 - 1\right), \quad M \in \mathbb{N}, \quad 1 \le k \le j.$$

It is well known (see [10, Lemma 2, p. 241] and [12, p.108])that

$$|Q'_{M,k}(x)| \le c_1 |x|^{-1} |1 - x^2|^{-1} Q_{M,k}(x), \quad \text{for } |x| \le 1,$$

and

$$Q_{M,k}(x) \sim \left(|x| + \frac{1}{M}\right)^{\gamma_k}, \quad \text{for } |x| \le 1.$$

Now for each $\epsilon \le n \in \mathbb{R}^+$, let $N = [n] + 1$ and

$$R_{n,k}(x) = (4Ba_n)^{\gamma_k} Q_{N,k}\left(\frac{x - x_k}{4Ba_n}\right), \quad \text{for } k = 1, 2, \cdots, j. \tag{2.7}$$

Then we have

$$|R'_{n,k}(x)| \le c_2 \frac{n}{a_n} R_{n,k}(x), \quad \text{for } x \in I_n \setminus A_{n,k}, \tag{2.8}$$

$$R_{n,k}(x) \sim \left(|x - x_k| + \frac{a_n}{n}\right)^{\gamma_k}, \quad \text{for } x \in I_n, \tag{2.9}$$

and

$$R_{n,k}(x) \sim v_k(x), \quad \text{for } x \in I_n \setminus A_{n,k}. \tag{2.10}$$

Now let

$$D_n = \Delta_n \setminus J_n \text{ and } B_{n,k} = A_{n,k} \cap \Delta_n.$$

Let $f \in \mathrm{GANP}_n$, $n \ge \epsilon$. First we show that

$$\int_{D_n} (fW)^p(x)dx \le c_3 \int_{I_n \setminus \Delta_n} (fW)^p(x)dx. \tag{2.11}$$

Since

$$R_{n,k}(x) \sim v_k(x), \quad x \in D_n, \quad 1 \le k \le i,$$

we have

$$\int_{D_n} (fW)^p(x)dx \le c_4 \int_{D_n} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x)dx.$$

Since $(fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m)$ is a generalized polynomial of degree less than $\Gamma_n = \mathcal{O}(n)$, by Lemma 2.1, (2.4), and (2.6), we obtain

$$\begin{aligned}
&\int_{D_n} (fW)^p(x)dx \\
&\quad \le c_5 \int_{I_n \setminus (\Delta_n \cup J_n)} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x)dx \\
&\quad \le c_6 \int_{I_n \setminus (\Delta_n \cup J_n)} (fW)^p(x)dx \\
&\quad \le c_6 \int_{I_n \setminus \Delta_n} (fW)^p(x).
\end{aligned}$$

Next we show that

$$\int_{B_{n,k}} (fW)^p(x)dx \le c_7 \int_{I_n \setminus \Delta_n} (fW)^p(x)dx, \quad 1 \le k \le m. \tag{2.12}$$

We distinguish two cases.

Case 1. $1 \le k \le i, (\gamma_k < 0)$. Since $(fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m)$ is a generalized polynomial of degree less than $\Gamma_n = \mathcal{O}(n)$, by Lemma 2.2, we have

$$\begin{aligned} &(fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x) \\ &\le c_8 \frac{n}{a_n} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(t)dt, \quad \text{for } x \in \mathbb{R}. \end{aligned} \tag{2.13}$$

Multiplying by $v_k^p(x)$ and then integrating both sides over $x \in A_{n,k}$, we obtain

$$\begin{aligned} &\int_{x \in A_{n,k}} (v_k f R_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x)dx \\ &\quad \le c_9 \left(\frac{a_n}{n}\right)^{p\gamma_k} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x)dx. \end{aligned}$$

Since

$$\left(\frac{n}{a_n}\right)^{\gamma_k} R_{n,k}(x) \ge c_{10}, \quad \text{for } x \in A_{n,k}, \quad \text{by (2.9)},$$

and

$$R_{n,\ell}(x) \sim v_\ell(x), 1 \le \ell \le i, \ell \ne k, \quad \text{for } x \in A_{n,k}, \quad \text{by (2.10)},$$

we have

$$\begin{aligned} &\int_{x \in A_{n,k}} (fW)^p(x)dx \\ &\quad \le c_{11} \int_{-\infty}^{\infty} (fR_{n,1} \cdots R_{n,i} v_{i+1} \cdots v_m W_\alpha)^p(x)dx. \end{aligned}$$

Then by Lemma 2.1, (2.4), and (2.6),

$$\begin{aligned}\int_{x\in A_{n,k}} &(fW)^p(x)dx\\ &\le c_{11}\int_{-\infty}^{\infty}(fR_{n,1}\cdots R_{n,i}v_{i+1}\cdots v_mW_\alpha)^p(x)dx\\ &\le c_{12}\int_{I_n\setminus(\Delta_n\cup J_n)}(fR_{n,1}\cdots R_{n,i}v_{i+1}\cdots v_mW_\alpha)^p(x)dx\\ &\le c_{13}\int_{I_n\setminus(\Delta_n\cup J_n)}(fW)^p(x)dx.\end{aligned}$$

Noting that

$$B_{n,k}\subset A_{n,k} \text{ and } I_n\setminus(\Delta_n\cup J_n)\subset I_n\setminus\Delta_n,$$

we have

$$\int_{x\in B_{n,k}}(fW)^p(x)dx\le c_{13}\int_{I_n\setminus\Delta_n}(fW)^p(x)dx,\quad 1\le k\le i. \tag{2.14}$$

Case 2. $i<k\le m,(\gamma_k\ge 0)$. Integrating both sides of (2.13) over $x\in A_{n,k}$, we obtain

$$\begin{aligned}\int_{x\in A_{n,k}}&(fR_{n,1}\cdots R_{n,i}v_{i+1}\cdots v_mW_\alpha)^p(x)dx\\ &\le c_{14}\int_{-\infty}^{\infty}(fR_{n,1}\cdots R_{n,i}v_{i+1}\cdots v_mW_\alpha)^p(x)dx.\end{aligned}$$

Since

$$R_{n,\ell}(x)\sim v_\ell(x), 1\le\ell\le i,\quad \text{for } x\in A_{n,k},\quad \text{by (2.10)},$$

we have

$$\begin{aligned}\int_{x\in A_{n,k}}&(fW)^p(x)dx\\ &\le c_{15}\int_{-\infty}^{\infty}(fR_{n,1}\cdots R_{n,i}v_{i+1}\cdots v_mW_\alpha)^p(x)dx.\end{aligned}$$

Then by Lemma 2.1, (2.4), and (2.5),

$$\begin{aligned}\int_{x\in A_{n,k}} &(fW)^p(x)dx\\ &\le c_{15}\int_{-\infty}^{\infty}(fR_{n,1}\cdots R_{n,i}v_{i+1}\cdots v_mW_\alpha)^p(x)dx\\ &\le c_{16}\int_{I_n\setminus(\Delta_n\cup J_n)}(fR_{n,1}\cdots R_{n,i}v_{i+1}\cdots v_mW_\alpha)^p(x)dx\\ &\le c_{17}\int_{I_n\setminus(\Delta_n\cup J_n)}(fW)^p(x)dx,\end{aligned}$$

hence,

$$\int_{x\in B_{n,k}}(fW)^p(x)dx\le c_{17}\int_{I_n\setminus\Delta_n}(fW)^p(x)dx,\quad i<k\le m.$$

Combining (2.11) and (2.12) yields

$$\int_{\Delta_n}(fW)^p(x)dx\le c_{18}\int_{I_n\setminus\Delta_n}(fW)^p(x)dx. \tag{2.15}$$

Next we show

$$\int_{|x|\ge Ba_n}(fW)^p(x)dx\le c_{19}\int_{I_n\setminus\Delta_n}(fW)^p(x)dx. \tag{2.16}$$

Let, for $1\le k\le i$,

$$M_{n,k}=\max(|Ba_n+x_k|^{p\gamma_k},|Ba_n-x_k|^{p\gamma_k})$$

and

$$m_{n,k}=\min(|Ba_n+x_k|^{p\gamma_k},|Ba_n-x_k|^{p\gamma_k}).$$

Then

$$\frac{M_{n,k}}{m_{n,k}}\le C(k),\quad \text{by (2.5)},$$

hence,

$$\begin{aligned}\int_{|x|\ge Ba_n}&(fW)^p(x)dx\\ &\le M_{n,1}\cdots M_{n,i}\int_{|x|\ge Ba_n}(fv_{i+1}\cdots v_mW_\alpha)^p(x)dx\\ &\le M_{n,1}\cdots M_{n,i}\int_{-\infty}^{\infty}(fv_{i+1}\cdots v_mW_\alpha)^p(x)dx,\end{aligned}$$

therefore, by Lemma 2.1,

$$\begin{aligned}
&\int_{|x|\geq Ba_n} (fW)^p(x)dx \\
&\qquad \leq c_{20} M_{n,1}\cdots M_{n,i} \int_{I_n\setminus\Delta_n} (fv_{i+1}\cdots v_m W_\alpha)^p(x)dx \\
&\qquad \leq c_{20} \frac{M_{n,1}\cdots M_{n,i}}{m_{n,1}\cdots m_{n,i}} \int_{I_n\setminus\Delta_n} (fW)^p(x)dx \\
&\qquad \leq c_{21} \int_{I_n\setminus\Delta_n} (fW)^p(x)dx.
\end{aligned}$$

Then by (2.15) and (2.16), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} (fW)^p(x)dx \\
&\quad = \int_{\Delta_n} (fW)^p(x)dx + \int_{I_n\setminus\Delta_n} (fW)^p(x)dx + \int_{|x|\geq Ba_n} (fW)^p(x)dx \\
&\quad \leq c_{22} \int_{I_n\setminus\Delta_n} (fW)^p(x)dx,
\end{aligned}$$

hence, Theorem 1.1 is proved. □

Proof of Corollary 1.2. Corollary 1.2 follows directly from Theorem 1.1. □

Proof of Theorem 1.3. Let $\epsilon > 0$, $d > 0$, and $0 < p \leq \infty$. For simplicity we consider

$$W_n(x) = \prod_{k=1}^{2} \left(|x - x_k| + \frac{a_n}{n}\right)^{\gamma_k} \cdot \exp(-|x|^\alpha),$$

where $n \in \mathbb{R}^+$, $\alpha > 1$, and

$$\gamma_1 < 0 \text{ and } \gamma_2 \geq 0.$$

General case follows by the same method. Let

$$\beta(n) = 5n + \gamma_2.$$

Let B^* be the constant which satisfies (2.1). Choose $B > 0$ big enough so that

$$B^* a_{\beta_n} \leq Ba_n, \quad \text{for } n \geq \epsilon, \tag{2.17}$$

and

$$|x_k| < (Ba_n)/2, \quad \text{for } k = 1, 2, \text{ and } n \geq \epsilon. \tag{2.18}$$

Let

$$I_n = [-Ba_n, Ba_n],$$

and let Δ_n be any measurable subset of I_n with $m(\Delta_n) \leq s_n$, where

$$s_n = \min\left\{\frac{da_n}{n}, a_n\right\}.$$

Let

$$v_{n,k}(x) = \left(|x - x_k| + \frac{a_n}{n}\right)^{\gamma_k}, \quad k = 1, 2.$$

And let

$$u_{n,1}(x) = \left|x - x_2 + \frac{a_n}{n}\right|^{\gamma_2}$$

and

$$u_{n,2}(x) = \left|x - x_2 - \frac{a_n}{n}\right|^{\gamma_2}.$$

We use the polynomial $R_{n,1}$ which we constructed in the proof of Theorem 1.1. See (2.7) and (2.9). Recall that $R_{n,1}$ has degree at most $4n$ and

$$R_{n,1}(x) \sim v_{n,1}(x), \quad x \in I_n. \tag{2.19}$$

Note that

$$\begin{aligned}
&\frac{1}{2}\left(\left|x - x_2 + \frac{a_n}{n}\right| + \left|x - x_2 - \frac{a_n}{n}\right|\right) \\
&\leq \left(|x - x_2| + \frac{a_n}{n}\right) \\
&\leq \left|x - x_2 + \frac{a_n}{n}\right| + \left|x - x_2 - \frac{a_n}{n}\right|, \quad x \in \mathbb{R}.
\end{aligned}$$

Then using

$$c_1(p)(|a| + |b|)^p \leq (|a|^p + |b|^p) \leq c_2(p)(|a| + |b|)^p, \quad (0 < p < \infty),$$

we have

$$v_{n,2}(x) \sim (u_{n,1}(x) + u_{n,2}(x)), \quad x \in \mathbb{R}. \tag{2.20}$$

Let $f \in \mathrm{GANP}_n$. Since $(fR_{n,1}u_{n,1})$ has degree at most $\beta(n) = \mathcal{O}(n)$, by Lemma 2.1 and (2.17), we have

$$||fR_{n,1}u_{n,1}W_\alpha||_{L^p(\Delta_n)} \leq c_1||fR_{n,1}u_{n,1}W_\alpha||_{L^p(I_n \setminus \Delta_n)}.$$

Since

$$R_{n,1}(x) \sim v_{n,1}(x), \quad x \in I_n,$$

and

$$u_{n,1}(x) \le v_{n,2}(x), \quad x \in \mathbb{R},$$

we have

$$||fR_{n,1}u_{n,1}W_\alpha||_{L^p(\Delta_n)} \le c_2||fW_n||_{L^p(I_n\setminus\Delta_n)}.$$

Similarly we obtain

$$||fR_{n,1}u_{n,2}W_\alpha||_{L^p(\Delta_n)} \le c_2||fW_n||_{L^p(I_n\setminus\Delta_n)}.$$

Then by (2.19) and (2.20),

$$\begin{aligned} ||fW_n||_{L^p(\Delta_n)} &= ||fv_{n,1}v_{n,2}W_\alpha||_{L^p(\Delta_n)} \\ &\le c_3||fR_{n,1}(u_{n,1}+u_{n,2})W_\alpha||_{L^p(\Delta_n)} \\ &\le c_4(||fR_{n,1}u_{n,1}W_\alpha||_{L^p(\Delta_n)} \\ &\quad +||fR_{n,1}u_{n,2}W_\alpha||_{L^p(\Delta_n)}) \\ &\le c_5||fW_n||_{L^p(I_n\setminus\Delta_n)}. \qquad (2.21) \end{aligned}$$

Next we show that

$$||fW_n||_{L^p(\mathbb{R}\setminus I_n)} \le c_6||fW_n||_{L^p(I_n\setminus\Delta_n)}.$$

Let

$$M_n = \max_{|x|\ge Ba_n}\left\{\left(|x-x_1|+\frac{a_n}{n}\right)^{\gamma_1}\right\}$$

and

$$m_n = \min_{|x|\le Ba_n}\left\{\left(|x-x_1|+\frac{a_n}{n}\right)^{\gamma_1}\right\}.$$

Then by (2.18)

$$\frac{M_n}{m_n} \le c_7,$$

hence, by Lemma 2.1 we have

$$\begin{aligned} ||fv_{n,1}u_{n,1}W_\alpha||_{L^p(\mathbb{R}\setminus I_n)} &\le M_n||fu_{n,1}W_\alpha||_{L^p(\mathbb{R}\setminus I_n)} \\ &\le M_n||fu_{n,1}W_\alpha||_{L^p(\mathbb{R})} \\ &\le c_8M_n||fu_{n,1}W_\alpha||_{L^p(I_n\setminus\Delta_n)} \\ &\le c_8\frac{M_n}{m_n}||fv_{n,1}u_{n,1}W_\alpha||_{L^p(I_n\setminus\Delta_n)} \\ &\le c_9||fv_{n,1}u_{n,1}W_\alpha||_{L^p(I_n\setminus\Delta_n)}. \end{aligned}$$

Since

$$u_{n,1}(x) \le v_{n,2}(x), \quad x \in \mathbb{R},$$

we obtain

$$||fv_{n,1}u_{n,1}W_\alpha||_{L^p(\mathbb{R}\setminus I_n)} \le c_9||fW_n||_{L^p(I_n\setminus\Delta_n)}.$$

Similarly we have

$$||fv_{n,1}u_{n,2}W_\alpha||_{L^p(\mathbb{R}\setminus I_n)} \leq c_9||fW_n||_{L^p(I_n\setminus\Delta_n)}.$$

Then by (2.20),

$$\begin{aligned} ||fW_n||_{L^p(\mathbb{R}\setminus I_n)} &= ||fv_{n,1}v_{n,2}W_\alpha||_{L^p(\mathbb{R}\setminus I_n)} \\ &\leq c_{10}||fv_{n,1}(u_{n,1}+u_{n,2})W_\alpha||_{L^p(\mathbb{R}\setminus I_n)} \\ &\leq c_{11}(||fv_{n,1}u_{n,1}W_\alpha||_{L^p(\mathbb{R}\setminus I_n)} \\ &\quad +||fv_{n,1}u_{n,2}W_\alpha||_{L^p(\mathbb{R}\setminus I_n)}) \\ &\leq c_{12}||fW_n||_{L^p(I_n\setminus\Delta_n)}. \end{aligned}$$

Combining (2.21) and the above inequality gives Theorem 1.3. □

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