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REDUCTION METHOD APPLIED TO THE NONLINEAR BIHARMONIC PROBLEM

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Abstract. We consider the semilinear biharmonic equation with Dirichlet boundary condition. We give a theorem that there exist at least three nontrivial solutions for the semilinear biharmonic boundary value problem. We show this result by using the critical point theory, the finite dimensional reduction method and the shape of the graph of the corresponding functional on the finite reduction subspace.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let Δ be the elliptic operator and Δ^2 be the biharmonic operator. In this paper we investigate the number of the weak solutions of the following semilinear biharmonic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = b(u+1)^+ - b \qquad \text{in } \Omega, \tag{1.1}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where $u^+ = \max\{u.0\}$ and b, $c \in R$. Choi and Jung [3] show that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega,, \tag{1.2}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

has at least two nontrivial solutions when ($c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$) or ($\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$). They obtained these results by use of the variational reduction method. They [5] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and

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$s < 0$, (1.2) has at least three nontrivial solutions by use of the degree theory. Tarantello [9] also studied (1.1). She show that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then (1.1) has a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [7] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$, then (1.1) has at least four solutions by the variational linking theorem and Leray-S chauder degree theory. In this paper we are looking for the weak solutions of (1.1) when

$$\lambda_k < c < \lambda_{k+1} \text{ and } \lambda_{k+m}(\lambda_{k+m} - c) < b < \lambda_{k+m+1}(\lambda_{k+m+1} - c).$$

The eigenvalue problem

$$\Delta^2 u + c\Delta u = \Lambda u \qquad \text{in } \Omega,$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

has infinitely many eigenvalues $\lambda_k(\lambda_{k-c})$, $k \geq 1$ and corresponding eigenfunctions ϕ_k, $k \geq 1$, the suitably normalized with respect to $L^2(\Omega)$ inner product, of where each eigenvalue λ_k is repeated as often as its multiplicity, where λ_k, $k \geq 1$, are the infinitely many eigenvalues and ϕ_k, $k \geq 1$, are the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product of the eigenvalue problem

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega.$$

We recall that $\lambda_1(\lambda_1 - c) \leq \lambda_2(\lambda_2 - c) \leq \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

Our main result is the following.

THEOREM 1.1. *Assume that* $\lambda_k < c < \lambda_{k+1}$ *and* $\lambda_{k+m}(\lambda_{k+m} - c) < b < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$. *Then (1.1) has at least three nontrivial solutions.*

The outline of the proof is as follows: In section 2, we show that the corresponding functional $I(u) \in C^1(H, R)$, *Fréchet* differentiable and satisfies the Palais-Smale condition. In section 3, we prove Theorem 1.1. For the proof of Theorem 1.1 we use the finite dimensional reduction method and investigate the (P.S.) condition and the critical points of the corresponding functional $\tilde{I}(v)$ on the finite reduction subspace.

2. Reduction method

Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \sum |\lambda_k(\lambda_k - c)| < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have
(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\||u|\| = 0$,
which is proved in [2].
We are looking for the weak solutions of (1.1). The weak solutions of (1.1) coincide with the critical points of the associated functional

$$I \in C^1(H, R),$$

$$I(u) = \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|(u+1)^+|^2 - bu]dx, \tag{2.1}$$

By the following Lemma 2.1, $I \in C^1(H, R)$ and I is *Fréchet* differentiable in H:

LEMMA 2.1. *Assume that* $\lambda_k < c < \lambda_{k+1}$ *and* $\lambda_{k+m}(\lambda_{k+m} - c) < b < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$. *Then* $I(u)$ *is continuous and Frechét differentiable in* H *and*

$$DI(u)(h) = \int_\Omega \Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - (b(u+1)^+ - b)h \tag{2.2}$$

for $h \in H$.

Proof. Let u, $v \in H$. Let us set $G(u) = \frac{b}{2}|(u+1)^+|^2 - bu$. First we will prove that $I(u)$ is continuous. We consider

$$
\begin{aligned}
&I(u+v) - I(u)\\
&= \int_\Omega [\frac{1}{2}|\Delta(u+v)|^2 - \frac{c}{2}|\nabla(u+v)|^2 - G(u+v)]\\
&- \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - G(u)]\\
&= \int_\Omega [u \cdot (\Delta^2 v + c\Delta v) + \frac{1}{2} v \cdot (\Delta^2 v + c\Delta v) - (G(u+v) - G(u))].
\end{aligned}
$$

Let $u = \sum h_k \phi_k$, $v = \sum \tilde{h}_k \phi_k$. Then we have

$$
|\int_\Omega u \cdot (\Delta^2 v + c\Delta v)dx| = |\sum \int_\Omega \lambda_k(\lambda_k - c) h_k \tilde{h}_k| \le \|u\|\|v\|,
$$

$$
|\int_\Omega v \cdot (\Delta^2 v + c\Delta v)dx| = |\sum \int_\Omega \lambda_k(\lambda_k - c) \tilde{h}_k^2| \le \|v\|^2.
$$

On the other hand, by Mean Value Theorem, we have

$$
\begin{aligned}
&\int_\Omega [G(u+v) - G(u)]\\
&= \int_\Omega [\frac{b}{2}|(u+v+1)^+|^2 - \frac{b}{2}|(u+1)^+|^2 - b(u+v) + bu]dx\\
&= \int_\Omega (b(u+tv+1)^+ - b)v dx\\
&\le b(\|(u+tv+1)^+\|_{L^2(\Omega)} - 1)\|v\|_{L^2(\Omega)}\\
&\le b(\|u+tv+1\|_{L^2(\Omega)} - 1)\|v\|_{L^2(\Omega)}\\
&\le b\|u+tv\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\\
&\le b\|u+v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\\
&\le b\|u+v\|\|v\| = O(\|v\|),
\end{aligned}
$$

where $0 < t < 1$. With the above results, we see that $I(u)$ is continuous at u. To prove $I(u)$ is *Fréchet* differentiable at $u \in H$, we consider

$$
\begin{aligned}
&|I(u+v) - I(u) - DI(u)v| \\
&= |\int_\Omega \frac{1}{2}v(\Delta^2 v + c\Delta v) - G(u+v) + G(u) - g(u)v| \\
&\leq \frac{1}{2}\|v\|^2 + b\|v\|(\|u\| + \|v\|) + b\|u\|\|v\| \\
&\leq (\frac{1}{2}\|v\| + b(2\|u\| + \|v\|))\|v\| = 0(\|v\|).
\end{aligned}
$$

Thus $I(u)$ is *Fréchet* differentiable at $u \in H$. □

Now we shall reduce the theory on the infinite dimensional space to the theory on the finite dimensional subspace. Let V be a m dimensional subspace of H which is the closure of span of the eigenfunctions whose corresponding eigenvalues are $\Lambda \leq \lambda_{k+m}(\lambda_{k+m} - c)$ and W be the orthogonal complement of V in H. Let $P : H \to V$ denote the orthogonal projection of H onto V and $I - P : H \to W$ denote that of H onto W. Then every element $u \in H$ is expressed by $u = v + w$, $v = Pu$, $w = (I - P)u$. Then (1.1) is equivalent to the two systems with two unknowns v and w:

$$
\begin{aligned}
\Delta^2 v + c\Delta v &= P(b(v + w + 1)^+ - b) \qquad \text{in } \Omega, && (2.3)\\
\Delta^2 w + c\Delta w &= (I - P)(b(v + w + 1)^+ - b) \qquad \text{in } \Omega, && (2.4)\\
v = 0, &\qquad \Delta v = 0 \qquad \text{on } \partial\Omega, \\
w = 0, &\qquad \Delta w = 0 \qquad \text{on } \partial\Omega.
\end{aligned}
$$

We recall that if I is a function of class C^1 and u_0 is a critical point of I, then u_0 is called of mountain pass type if for every open neighborhood U of $I^{-1}(-\infty, I(u_0)) \cap U \neq \emptyset$ and $I^{-1}(-\infty, I(u_0)) \cap U$ is not pass connected.

Lemma 2.2. *(Finite Dimensional Reduction Lemma) Let $\lambda_k < c < \lambda_{k+1}$. Assume that $\lambda_{k+m}(\lambda_{k+m} - c) < b < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $k \geq 1$, $m \geq 1$. Then we have that*
(i) there exists $m > 0$ such that

$$(DI(v+w) - DI(v+w_1), w - w_1) \geq m\|w - w_1\|^2, \text{for all } v \in V, w, w_1 \in W.$$

(ii) there exists a unique solution $w \in W$ of the form

$$
\begin{aligned}
\Delta^2 w + c\Delta w - (I - P)(b(v + w + 1)^+ - b) = 0 \qquad \text{in } W, && (2.5)\\
w = 0, \qquad \Delta w = 0 \qquad \text{on } \partial\Omega.
\end{aligned}
$$

If we put $w=\theta(v)$, then θ is continuous on V and satisfies a uniform Lipschitz condition in v with respect to the L^2 norm(also norm $\|\cdot\|$). Moreover

$$\nabla I(v+\theta(v))(w)=0 \qquad \text{for all } w\in W. \tag{2.6}$$

(iii) If $\tilde{I}:V\to R$ is defined by $\tilde{I}(v)=I(v+\theta(v))$, then $\tilde{I}$ has a continuous Fréchet derivative $\nabla\tilde{I}$ with respect to v, and

$$\nabla\tilde{I}(v)(h)=\nabla I(v+\theta(v))(h) \qquad \text{for all } v,h\in V.$$

(iii) If $v_0\in V$ is a critical point of $\tilde{I}$ if and only if $v_0+\theta(v_0)$ is a critical point of I.
(iv) If $u_0=v_0+\theta(v_0)$ is a critical point of mountain pass type of I, then v_0 is a critical point of mountain pass type of $\tilde{I}$.

Proof. (i)If $v\in V$, $w,w_1\in W$, then

$$(DI(v+w)-DI(v+w_1))(w-w_1)=\int_\Omega[|\Delta(w-w_1)|^2-c|\nabla(w-w_1)|^2$$
$$-(b(v+w+1)^+-b(v+w_1+1)^+)(w-w_1)]$$

According to the variational characterization of the eigenvalues $\{\lambda_j(\lambda_j-c)\}_1^\infty$, we have $\|w\|^2\ge\lambda_{k+m+1}(\lambda_{k+m+1}-c)\|w\|_{L^2(\Omega)}$. Since $(b(v+w+1)^+-b(v+w_1+1)^+)(w-w_1)\le b(w-w_1)^2$, it follows that

$$\begin{aligned}&(DI(v+w)-DI(v+w_1))(w-w_1)\\&\qquad\ge\|w-w_1\|^2-b\|w-w_1\|^2_{L^2(\Omega)}\\&\qquad=m\|w-w_1\|^2\end{aligned}$$

with $m=1-\frac{b}{\lambda_{k+m+1}(\lambda_{k+m+1}-c)}>0$. Thus (i) is satisfied.
(ii)The equation (2.5) is equivalent to

$$w=(\Delta^2+c\Delta-\frac{b}{2})^{-1}(I-P)(b(v+w+1)^+-b-\frac{b}{2}(v+w)) \tag{2.7}$$

Since $(\Delta^2+c\Delta-\frac{b}{2})^{-1}(I-P)$ is self adjoint, compact and linear map from $(I-P)L^2(\Omega)$ into itself, the eigenvalues of $(\Delta^2+c\Delta-\frac{b}{2})^{-1}(I-P)$ are $(\lambda_l(\lambda_l-c)-\frac{b}{2})^{-1}$, $l\ge k+m+1$. Therefore its L_2 norm is $(\lambda_{k+m+1}(\lambda_{k+m+1}-c)-\frac{b}{2})^{-1}$. Since

$$\begin{aligned}&\{b(v+w+1)^+-b-\frac{b}{2}(v+w)\}-\{b(v'+w'+1)-b+\frac{b}{2}(v'+w')\}\\ \le\ &(b-\frac{b}{2})|(v+w+1)-(v'+w'+1)|,\end{aligned}$$

it follows that the right-hand side of (2.7) defines, for fixed $v \in V$, a Lipschitz mapping of $(I-P)L^2(\Omega)$ into itself with Lipschitz constant $r = \frac{1}{\lambda_{k+m+1}(\lambda_{k+m+1}-c)-\frac{b}{2}} \cdot (b - \frac{b}{2}) < 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $w = (I-P)L^2(\Omega)$ which satisfies (2.5). If $\theta(v)$ denote the unique $w \in (I-P)L^2(\Omega)$ which solves (2.5), then θ is continuous and satisfies a uniform Lipschitz condition in v with respect to the L^2 norm(also norm $\|\cdot\|$). In fact, if $w_1 = \theta(v_1)$ and $w_2 = \theta(v_2)$, then

$$\begin{aligned}
&\|w_1 - w_2\| \\
&= \|(\Delta^2 + c\Delta - \frac{b}{2})(I-P)(\{b(v_1+w_1+1)^+ - b - \frac{b}{2}(v_1+w_1)\} \\
&- \{b(v_2+w_2+1) - b + \frac{b}{2}(v_2+w_2)\}\| \\
&\le \|(\Delta^2 + c\Delta - \frac{b}{2})(I-P)(\{b(v_1+w_1+1)^+ - b - \frac{b}{2}(v_1+w_1)\} \\
&- \{b(v_2+w_2+1) - b + \frac{b}{2}(v_2+w_2)\}\| \\
&= r\|(v_1+w_1) - (v_2+w_2)\| \\
&\le r(\|v_1 - v_2\| + \|w_1 - w_2\|) \le r|\|v_1 - v_2\|| + r|\|w_1 - w_2\||.
\end{aligned}$$

Hence

$$|\|w_1 - w_2\|| \le C|\|v_1 - v_2\||, \qquad C = \frac{r}{1-r}. \tag{2.8}$$

Let $u = v + w$, $v \in V$ and $w = \theta(v)$. If $w \in (I-P)L^2(\Omega) \cap H$, then from (2.5) we see that

$$\int_\Omega [\Delta w \cdot \Delta z - c\nabla w \cdot \nabla z - (I-P)(b(v+w+1)^+ z - bz)]dx = 0.$$

Since

$$\int_\Omega \Delta w \cdot \Delta z = 0 \qquad \text{and} \quad \int_\Omega \nabla v \cdot \nabla z = 0,$$

we have

$$DI(v + \theta(v))(w) = 0. \tag{2.9}$$

(iii) Since the functional I has a continuous *Fréchet* derivative DI, $\tilde{I}$ has a continuous *Fréchet* derivative $D\tilde{I}$ with respect to v.

(iv) Suppose that there exists $v_0 \in V$ such that $D\tilde{I}(v_0) = 0$. From $D\tilde{I}(v)(h) = DI(v+\theta(v))(h)$ for all $v, h \in V$, $DI(v_0 + \theta(v_0))(h) = 0$ for all $h \in V$. Since $DI(v+\theta(v))(w)$ for all $w \in W$ and H is the direct sum

of V and W, it follows that $DI(v_0 + \theta(v_0)) = 0$. Thus $v_0 + \theta(v_0)$ is a solution of (1.1). Conversely if u is a solution of (1.1) and $v = Pu$, then $D\tilde{I}(v) = 0$.
(iv) Suppose v_0 is not of mountain pass type of $\tilde{I}$. Let S be an open neighborhood of v_0 in V such that $\tilde{I}^{-1}(-\infty, \tilde{I}(v_0)) \cap S$ is empty or path connected. If $\tilde{I}^{-1}(-\infty, \tilde{I}(v_0)) \cap S$ is empty, by part (i) we see that $\{v + w : v \in V, w \in W\} \cap I^{-1}(-\infty, I(u_0))$ is also empty. Thus u_0 is not of mountain pass type for I. If $\tilde{I}^{-1}(-\infty, \tilde{I}(v_0)) \cap S$ is path connected, Letting $T = \{v + w : v \in V, \|w - \theta(v)\| < 1\}$ and using again (i) it is seen that $T \cap I^{-1}(-\infty, I(u_0))$ is also path connected. □

3. Palais-Smale condition and proof of theorem 1.1

We shall show that $\tilde{I}(v)$ satisfies the (P.S.) condition.

LEMMA 3.1. *Assume that* $\lambda_k < c < \lambda_{k+1}$ *and* $\lambda_{k+m}(\lambda_{k+m} - c) < b < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$. *Then* $\tilde{I}(v)$ *satisfies the Palais-Smale condition.*

Proof. Let us set $u(v) = v + w(v)$, $v \in V$, $w(v) \in W$. Then we have

$$\tilde{I}(v) = \int_\Omega [\frac{1}{2}|\Delta v + \Delta w(v)|^2 - \frac{c}{2}|\nabla v + \nabla w(v)|^2]dx$$

$$- \int_\Omega \frac{b}{2}|(v + w(v) + 1)^+|^2 dx + \int_\Omega b(v + w)dx.$$

Let us set $G(u(v)) = \int_\Omega [b(u(v) + 1)^+ - b]dx$. Then we have

$$\begin{aligned}
\tilde{I}(v) &= I(v + w(v)) = I(u(v)) \\
&= \int_\Omega [\frac{1}{2}|\Delta u(v)|^2 \frac{c}{2}|\nabla u(v)|^2 - b\int_\Omega G(u(v))dx \\
&= \int_\Omega [\frac{1}{2}|\Delta v|^2 - \frac{c}{2}|\nabla v|^2]dx - b\int_\Omega G(v)dx \\
&\quad +\{\int_\Omega \frac{1}{2}|\Delta u(v)|^2 - \frac{c}{2}|\nabla u(v)|^2 - \frac{1}{2}|\Delta v|^2 + \frac{c}{2}|\nabla v|^2 \\
&\quad -b\int_\Omega [G(u(v)) - G(v)]dx\}.
\end{aligned}$$

The terms in the bracket are equal to

$$-b\int_\Omega [G'(sw(v)-v)w(v)dx]ds + \frac{1}{2}\int_\Omega (\Delta^2 u(v) + c\Delta u(v))w(v)dx$$
$$= b\int_\Omega \int_0^1 G''(sw(v)+v)w(v)w(v)sdsdx$$
$$-\frac{1}{2}\int_\Omega (\Delta^2 w(v) + c\Delta w(v))w(v)dx$$

by the condition $\lambda_{k+m}(\lambda_{k+m}-c) < b < \lambda_{k+m+1}(\lambda_{k+m+1}-c)$. Thus we have

$$\begin{aligned}\tilde{I}(v) \quad \leq \quad & \int_\Omega [\frac{1}{2}|\Delta v|^2 - \frac{c}{2}|\nabla v|^2]dx \\ & -b\int_\Omega G(v)dx \\ & \frac{1}{2}\{\lambda_{k+m}(\lambda_{k+m}-c)-b\}\|v\|^2 = b\|v\| \longrightarrow -\infty \text{ as } \ \|v\| \to \infty.\end{aligned}$$

Thus $-\tilde{I}(v)$ is bounded from below and, so satisfies the (P.S.) condition.
□

LEMMA 3.2. *Under the same assumption as Theorem 1.1, 0 is neither a minimum nor degenerate.*

Proof. We note that $0 = 0 + \theta(0)$, $\theta(0) = 0$. Since $I + \theta$ is continuous, I is identity map, there exists a small neighborhood B of 0 such that if $v \in B$, then

$$\frac{1}{2}\int_\Omega (\Delta^2 v + c\Delta v)vdx - \frac{\bar{\Lambda}}{2}\int_\Omega G(v)dx + o(\|v\|^2) \leq \tilde{I}(v)$$
$$\leq \frac{1}{2}\int_\Omega (\Delta^2 v + c\Delta v)vdx - \frac{\Lambda}{2}\int_\Omega G(v)dx + o(\|v\|^2),$$

where $(\Lambda.\bar{\Lambda}) \subset (\lambda_{k+m}(\lambda_{k+m}-c), \lambda_{k+m+1}(\lambda_{k+m+1}-c))$.

□

Proof of theorem 1.1 By Lemma 3.1, $I(v)$ is bounded above, satisfies the (P.S.) condition and $\tilde{I}(v) \to -\infty$ as $\|v\| \to \infty$. By Lemma 3.3, 0 is neither a minimum nor degenerate. Thus $\tilde{I}(v)$ has at least three nontrivial weak solutions.

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