Kangweon–Kyungki Math. Jour. 1 (1993), pp. 7–19

THE MANN ITERATIVE PROCESS FOR QUASI–CONTRACTIVE MAPS IN UNIFORMLY SMOOTH BANACH SPACES

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1. Introduction.

If K is a nonempty subset of a normed linear space, and T is a map of K into itself, then T is called quasi-contractive [8] if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\| \leq k \max \{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\}$$

for all x, y in K.

Mann iteration process (see e.g. [4]) : the sequence $\{x_n\}$ is defined as follows:

(1) $x_0 \in K$,

(2) $x_{n+1} = (1-c_n)x_n + c_n T x_n, n \geq 0$,

where (i) $0 \leq c_n < 1$; (ii) $\lim_{n\to\infty} c_n = 0$; (iii) $\sum_{n=0}^{\infty} c_n = \infty$.

Oihou [6] considered the Ishikawa iterations, which are more complex than the Mann iterations, for the approximations to a unique fixed point of the quasi-contractive maps in Hilbert spaces. Recently Chidume and Osilike [1] obtained the generalizations of Oihou in uniformly smooth Banach spaces which have modulus of smoothness of power type $q > 1$. Weng [10] applied the Mann iteration to the map which satisfies the dissipative type condition in uniformly smooth Banach spaces.

In this paper we consider the same problem for quasi-contractive maps in uniformly smooth Banach spaces. Reich [7] obtained an inequality for the norm by using β-function in uniformly smooth Banach spaces. By using this inequality we have the lemma about the inequalities for the norm in uniformly smooth Banach spaces and the following conclusion; On some conditions Mann iterations converge to the fixed point of a quasi-contractive map in uniformly smooth Banach spaces.

This paper is the thesis for the Degree of Master in Kangweon National University (1992) .

2. Convexity and smoothness.

In this section we connect the smoothness of the norm of a Banach space X and the convexity of the norm of the dual of X which is denoted by X^*.

The set of real numbers is denoted by R. And the unit sphere (the unite solid ball) of X and X^* are denoted by $S(X)(B(X))$ and $S(X^*)(B(X^*))$ respectively.

$\phi : X \to R$ is said to be convex if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y),$$

where $0 \leq \lambda \leq 1$.

THEOREM 1 [9, p4]. *If $\phi : I \subset R \to R$ is convex, then ϕ is a Lipschitz function on any closed interval in the interior of a interval I.*

Proof. First we claim that ϕ is bounded below on the closed interval $[a,b]$ in the interior of I. Clearly ϕ is bounded above on $[a,b]$ by $M = \max\{|\phi(a)|, |\phi(b)|\}$. Hence

$$\phi(\frac{a+b}{2}) \leq \frac{1}{2}\phi(\frac{a+b}{2}+t) + \frac{1}{2}\phi(\frac{a+b}{2}-t), |t| \leq \frac{b-a}{2}.$$

Therefore

$$\phi(\frac{a+b}{2}+t) \geq 2\phi(\frac{a+b}{2}) - \phi(\frac{a+b}{2}-t) \geq 2\phi(\frac{a+b}{2}) - M.$$

So we conclude that ϕ is bounded on closed intervals. For any closed interval $[a,b] \subset I^\circ$, which is the interior of I we can choose $\epsilon > 0$ such that $a - \epsilon \in I$ and $b + \epsilon \in I$. Then ϕ is bounded on $[a-\epsilon, b+\epsilon]$. For any $x, y(\neq)$ in $[a,b]$ we let

$$z = y + \frac{\epsilon}{|y-x|}(y-x), \lambda = \frac{|y-x|}{\epsilon + |y-x|}.$$

Then $z \in [a-\epsilon, b+\epsilon]$ and $y = \lambda z + (1-\lambda)x$. So $\phi(y) \leq \lambda(\phi(z) - \phi(x)) + \phi(x)$. From this

$$\begin{aligned}\phi(y) - \phi(x) &\leq \lambda(M-m)\\ &< \frac{|y-x|}{\epsilon}(M-m)\\ &\leq K|y-x|,\end{aligned}$$

where M (and m) are the upper bound (lower bound) of ϕ on $[a-\epsilon, b+\epsilon]$ and $K=(M-m)/\epsilon$. Hence ϕ is a Lipschitz function on $[a,b]$. □

THEOREM 2 [5, p37]. *If $\phi : X \to R$ is convex, where X is a normed linear space, then $\psi(t) = (\phi(x+ty)-\phi(x))/t (t \neq 0)$ is nondecreasing.*

Proof. Case 1) $0 < h < k$. We can choose $\lambda \in (0,1)$ such that $h = (1-\lambda)k$. Then

$$\begin{aligned}\phi(x+hy) &= \phi(x+(1-\lambda)ky)\\ &= \phi(\lambda x + (1-\lambda)(x+ky))\\ &\le \lambda\phi(x) + (1-\lambda)\phi(x+ky).\end{aligned}$$

Therefore

$$\phi(x+hy)-\phi(x) \le (1-\lambda)(\phi(x+ky)-\phi(x))$$

Hence

$$\psi(h) = \frac{\phi(x+hy)-\phi(x)}{h} \le \frac{\phi(x+ky)-\phi(x)}{k} = \psi(k)$$

Case 2) $h < k < 0$. By choosing $\lambda \in (0,1)$ such that $k = (1-\lambda)h$ we obtain $\psi(h) \le \psi(k)$.

Case 3) $h < 0 < k$. We can choose $l > 0$ such that $h < -l < 0 < l < k$. Then

$$\begin{aligned}\phi(x) &= \phi(\frac{1}{2}(x-ly)+\frac{1}{2}(x+ly))\\ &\le \frac{1}{2}\phi(x-ly)+\frac{1}{2}\phi(x+ly)\end{aligned}$$

Hence

$$(\phi(x)-\phi(x-ly))/l \le (\phi(x+ly)-\phi(x))/l$$

Therefore

$$\psi(-l) \le \psi(l)$$

And by Cases 1), 2) we have

$$\psi(h) \le \psi(k).$$

□

Example 1. Let $\phi(x) = \|x\|$. By Theorem 2

$$\lim_{t\to 0+} \frac{\|x+ty\| - \|x\|}{t} \text{ and } \lim_{t\to 0^-} \frac{\|x+ty\| - \|x\|}{t} \text{ exist}$$

for all $x, y \in S(X)$. If the two limits are equal, then norm $\|\cdot\|$ is said to be Gâteaux differentiable.

Example 2. If the norm $\|\cdot\|$ is Gâteaux differentiable, then

$$\lim_{t\to 0} \frac{\|x+ty\|^2 - \|x\|^2}{t} = (\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t})2\|x\|.$$

The duality map $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* | < x, x^* >= \|x\|^2, \|x^*\| = \|x\|\}.$$

By Hahn-Banach theorem $J(x) \neq \emptyset$ for any $x \in X$. The Banach space X is said to be strictly convex if for any $x, y \in S(X)(\neq)$

$$\|\frac{x+y}{2}\| < 1.$$

The Banach space X is said to be uniformly convex if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $\|x - y\| \geq \epsilon, x, y \in S(X)$, then

$$\|x + y\| \leq 2(1 - \delta(\epsilon)).$$

Clearly the uniformly convex Banach spaces are strictly convex.

LEMMA 1 [2, p20]. *For any choice $j(x) \in J(x)$ and $\lambda > 0, x, y \in S(X)$,*

$$< y, j(x) >\leq \frac{\|x+\lambda y\| - \|x\|}{\lambda} \leq \frac{< y, j(x+\lambda y) >}{\|x+\lambda y\|}.$$

Proof.

$$\begin{aligned} \lambda < y, j(x) > &=< \lambda y, j(x) > \\ &=< x + \lambda y, j(x) > - \|x\| \\ &\leq \|x + \lambda y\| \|j(x)\| - \|x\| \\ &= \|x + \lambda y\| - \|x\| \end{aligned}$$

$$\begin{aligned}\lambda < y, j(x+\lambda y) > &= < x+\lambda y, j(x+\lambda y) > - < x, j(x+\lambda y) > \\ &\geq \|x+\lambda y\|^2 - \|x\|\|x+\lambda y\| \\ &= \|x+\lambda y\|(\|x+\lambda y\| - \|x\|).\end{aligned}$$

□

Clearly we know that if X^* is strictly convex, then J is single-valued. We obtain the following converse if X is reflexive.

LEMMA 2 [2, p24]. *If a Banach space X is reflexive and J is single-valued, then X^* is strictly convex.*

Proof. Suppose we have $\|(j_1 + j_2)/2\| = 1$ for some $j_1, j_2 \in S(X^*)$. Then, by James' theorem there exists $x \in S(X)$ such that

$$< x, \frac{j_1 + j_2}{2} >= 1.$$

Therefore

$$1 =< x, \frac{j_1}{2} > + < x, \frac{j_2}{2} > .$$

On the other hand, $< x, \frac{j_1}{2} >\leq \frac{1}{2}$ and $< x, \frac{j_2}{2} >\leq \frac{1}{2}$. Hence $< x, j_1 >= 1$ and $< x, j_2 >= 1$. Since J is single-valued and $j_1, j_2 \in J(x), j_1 = j_2$.
□

THEOREM 3 [2, p22]. *The followings are equivalent:*

(1) *J is singule-valued ;*
(2) *J is norm to weak-star continuous from $S(X)$ to $S(X^*)$;*
(3) *The norm of X is Gâteaux differentiable on $S(X)$.*

Proof. (1) $\Rightarrow$ (2) Suppose $\{x_n\}$ converges strongly to x in $S(X)$. Then $\{J(x_n)\} \subset S(X^*)$. Since the unit solid ball of X^* is weak-star compact by Banach-Alaoglu's Theorem, we can choose a subsequence $\{J(x_{n_k})\}$ which converges weak-star to $j \in X^*$ and $\|j\| \leq 1$. Therefore $< x_{n_k} - x, J(x_{n_k}) >\to 0$. So $1- < x, J(x_{n_k}) >\to 0$. Hence $< x, j >= 1$ and $\|j\| = 1$. By (1) $j = J(x)$.

(2) $\Rightarrow$ (3) By Lemma 1 it is clear.

(3) $\Rightarrow$ (1) By Lemma 1 we have, for $j \in J(x)$

$$\frac{\|x + ly\| - \|x\|}{l} \leq < y, j > \leq \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

for $l < 0, \lambda > 0$. By (3)

$$(y, j) = \lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

Hence, for any $j_1, j_2 \in J(x)$, we have $j_1 = j_2$ on $S(X)$. So $j_1 = j_2$. □

THEOREM 4 [3]. *If the dual X^* of a Banach space X is uniformly convex, then the duality map $J : X \to X^*$ is uniformly continuous on any bounded subset of X.*

Proof. Suppose on the contrary that there exist $M(> 0), \epsilon(> 0)$, $\{u_n\}$, $\{v_n\}$ in X such that $\|u_n\| \leq M, \|v_n\| \leq M$ and $\|u_n - v_n\| \to 0$ and

$$\|J(u_n) - J(v_n)\| \geq \epsilon.$$

We may assume that $u_n \neq 0$ and $v_n \neq 0$. Let $x_n = u_n/\|u_n\|$ and $y_n = v_n/\|v_n\|$. Then $\|x_n - y_n\| \to 0$. On the other hand

$$\begin{aligned} < x_n, J(x_n) + J(y_n) > &= \|x_n\|^2 + < x_n, J(y_n) > \\ &= \|x_n\|^2 + \|y_n\|^2 + < x_n - y_n, J(y_n) > \\ &\geq 2 - \|x_n - y_n\| \end{aligned}$$

Hence

$$< x_n, \frac{1}{2}(J(x_n) + J(y_n)) > \geq 1 - \frac{1}{2}\|x_n - y_n\|$$

So

$$\|\frac{1}{2}(J(x_n) + J(y_n))\| \geq 1 - \frac{1}{2}\|x_n - y_n\|$$

Since X^* is uniformly convex, $J(x_n) - J(y_n)$ converges strongly to 0 in X^*. Hence

$$\begin{aligned} J(u_n) - J(v_n) = &\|u_n\|(J(x_n) - J(y_n)) \\ &+ (\|u_n\| - \|v_n\|)J(y_n), \end{aligned}$$

Since $\{u_n\}$ is bounded and $\|u_n\| - \|v_n\| \to 0$, we obtain a contradiction.

We say that the norm of a Banach space X is uniformly Fréchet differentiable (away from 0) whenever

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

exists uniformly for $x, y \in S(X)$. □

THEOREM 5 [2, p36]. *The followings are equivalent:*

(1) *The dual X^* of a Banach space X is uniformly convex;*
(2) *The norm of X is uniformly Fréchet differentiable.*

Proof. (1)⇒(2). By Lemma 1,

$$\left|\frac{\|x+\lambda y\|-\|x\|}{\lambda} - < y, J(x) >\right| \le < y, \frac{J(x+\lambda y)}{\|x+\lambda y\|} - J(x) > .$$

By Theorem 4 J is uniformly continuous on $S(X)$. Hence for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $\|x' - y'\| < \delta(\epsilon), x', y' \in S(X)$, then $\|J(x') - J(y')\| < \epsilon$. So we choose $\delta'(> 0,$ depends on $\delta(\epsilon))$ such that if $0 < |\lambda| < \delta'$, then

$$\left\|\frac{x+\lambda y}{\|x+\lambda y\|} - x\right\| < \delta(\epsilon).$$

Then

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

converges to $< y, J(x) >$ uniformly for $x, y \in S(X)$.

(2)⇒(1). Since the norm of X is uniformly Fréchet differentiable, for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for any $x, y \in S(X)$ and $\lambda > 0(\lambda < \delta(\epsilon))$

$$g(x,y,\lambda) + g(x,y,-\lambda) < \frac{\epsilon}{2},$$

where

$$g(x,y,\lambda) = \frac{\|x+\lambda y\| - \|x\| - < \lambda y, J(x) >}{\|\lambda y\|}.$$

Hence

$$\|x+\lambda y\| + \|x-\lambda y\| < 2\|x\| + \epsilon\|\lambda y\|.$$

Therefore there exists $\delta' > 0$ (depends on ϵ) such that for any $x \in S(X)$ and $y(\|y\| < \delta')$,

$$\|x+y\| + \|x-y\| < 2 + \frac{\epsilon}{4}\|y\|.$$

For any $f, g \in S(X^*), \|f-g\| \geq \epsilon$, we choose $y \in X, \|y\| = \frac{\delta'}{2}$ such that

$$< y, f-g > \geq \frac{\epsilon\delta'}{4}.$$

Then

$$\begin{aligned}
\|f+g\| &= \sup_{x \in S(X)} < x, f+g > \\
&= \sup_{x \in S(X)} < x+y, f > + < x-y, g > - < y, f-g > \\
&< 2 + \frac{\epsilon}{4}\frac{\delta'}{2} - \frac{\epsilon}{4}\delta' \\
&= 2(1 - \frac{\epsilon}{16}\delta').
\end{aligned}$$

□

A Banach space X is said to be uniformly smooth if X^* is uniformly convex. We define for positive t

$$\beta(t) = \sup \{\frac{\|x+ty\|^2 - \|x\|^2}{t} - 2 < y, J(x) >: \|x\| \leq 1, \|y\| \leq 1\}.$$

By Theorem 2 $\beta(t)$ is nondecreasing and $\sup_{x,y \in S(X)} \|x+ty\|^2 - \|x\|^2$ is convex. Therefore by Theorem 1 $\beta(t)$ is continuous. And for $c \geq 1$ $\beta(ct) \leq c\beta(t)$.

THEOREM 6 [7]. *If X is uniformly smooth Banach space, then* $\lim_{t \to 0^+} \beta(t) = 0$ *and*

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, J(x) > + \max \{\|x\|, 1\}\|y\|\beta(\|y\|)$$

for all $x, y \in X$.

Proof. For $t \geq 0$, let

$$\alpha(t) = \sup \{\|J(x) - J(y)\| : \|x\| \leq 2, \|y\| \leq 2, \|x-y\| \leq t\}.$$

Since X^* is uniformly convex, by Theorem 4

$$\lim_{t \to 0^+} \alpha(t) = 0.$$

Let $F(s) = \|x + sy\|^2/2$. Then

$$F(s+h) - F(s) = \frac{1}{2}(\|x + (s+h)y\|^2 - \|x + sy\|^2)$$
$$= \frac{1}{2}(\|x + sy + hy\| - \|x + sy\|)\cdot$$
$$(\|x + sy + hy\| + \|x + sy\|)$$

Hence

$$F'(s) = \lim_{h\to 0} \|x + sy\|(\frac{\|x + sy + hy\| - \|x + sy\|}{h})$$
$$=< y, J(x + sy) > .$$

Therefore

$$F(t) - F(0) = \int_0^t < y, J(x + sy) > ds$$

Hence for $x, y \in B(X)$

$$\frac{\|x + ty\|^2 - \|x\|^2}{t} - 2 < y, J(x) >= \frac{2}{t}\int_0^t < y, J(x + sy) - J(x) > ds.$$

For $t \leq 1$

$$\beta(t) \leq \frac{2}{t} \cdot t\alpha(t) = 2\alpha(t)$$

Hence $\lim_{t\to 0+} \beta(t) = 0$. So we define $\beta(0) = 0$. For $t \geq 0, x, y \in B(X)$

$$\|x + ty\|^2 - \|x\|^2 - 2 < ty, J(x) >\leq t\beta(t).$$

Hence for any $x \in B(X), y \in X$,

$$\|x + y\|^2 - \|x\|^2 - 2 < y, J(x) >\leq \|y\|\beta(\|y\|).$$

If $\|x\| \geq 1$, then

$$\|x + y\|^2 - \|x\|^2 - 2 < y, J(x) >\leq \|x\|\|y\|\beta(\|y\|).$$

Hence

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, J(x) > + \max\{\|x\|, 1\}\|y\|\beta(\|y\|).$$

□

LEMMA 3 [10]. *Let* $\{\beta_n\}$ *be a nonnegative sequence satisfy*

$$\beta_{n+1} \leq (1-\delta_n)\beta_n + \sigma_n$$

with

$$\delta_n \in [0,1], \sum_{i=1}^{\infty} \delta_i = \infty$$

and

$$\lim_{n\to\infty} \frac{\sigma_n}{\delta_n} = 0.$$

Then

$$\lim_{n\to\infty} \beta_n = 0$$

Proof. Let $\frac{\sigma_n}{\delta_n} = \epsilon_n$. Then

$$\begin{aligned}\beta_{n+1} &\leq (1-\delta_n)((1-\delta_{n-1})\beta_{n-1} + \sigma_{n-1}) + \sigma_n \\ &= (1-\delta_n)(1-\delta_{n-1})\beta_{n-1} + (1-\delta_n)\sigma_{n-1} + \sigma_n \\ &\leq (1-\delta_n)(1-\delta_{n-1})(1-\delta_{n-2})\beta_{n-2} \\ &+ (1-\delta_n)(1-\delta_{n-1})\sigma_{n-2} + (1-\delta_n)\sigma_{n-1} + \sigma_n.\end{aligned}$$

By induction

$$\beta_{n+1} \leq [\Pi_{j=k}^{n}(1-\delta_j)]\beta_k + \sum_{j=k}^{n}(\Pi_{i=j+1}^{n}(1-\delta_i)) \cdot \sigma_j.$$

So

$$\beta_{n+1} \leq [\Pi_{j=k}^{n}(1-\delta_j)]\beta_k + \sum_{j=k}^{n}(\delta_j \Pi_{i=j+1}^{n}(1-\delta_i)) \cdot \epsilon_j.$$

Since $1-\delta_j \leq \exp(-\delta_j)$ for any j,

$$\Pi_{j=k}^{n}(1-\delta_j) \leq \exp(-\sum_{j=k}^{n}\delta_j) \cdots\cdots (*).$$

And we prove by induction that

$$\sum_{j=k}^{n}(\delta_j \Pi_{i=j+1}^{n}(1-\delta_i)) \leq 1 \cdots\cdots (**).$$

If $n = k+1$, then $\delta_{n-1}(1-\delta_n) + \delta_n \leq 1$. If the above inequality $(**)$ holds for n, then we let $A_n = \sum_{j=k}^{n} \delta_j(1-\delta_{j+1})\cdots(1-\delta_n)$. By induction hypothesis $A_n \leq 1$. Then

$$\begin{aligned} A_{n+1} &= \sum_{j=k}^{n+1} \delta_j(1-\delta_{j+1})\cdots(1-\delta_{n+1}) \\ &= \sum_{j=k}^{n} \delta_j(1-\delta_{j+1})\cdots(1-\delta_{n+1}) + \delta_{n+1} \\ &= (1-\delta_{n+1})A_n + \delta_{n+1} \\ &\leq 1. \end{aligned}$$

Hence for any $\epsilon \geq 0$ there exists k such that $\epsilon_n \leq \epsilon$ for $n \geq k$. Hence for sufficiently large n

$$0 \leq \beta_{n+1} \leq \epsilon.$$

□

3. Main results.

We obtain the following inequality in uniformly smooth Banach space.

LEMMA 4. *Let X be uniformly smooth Banach space. Let β be given in Theorem 6. Then*

$$\begin{aligned} \|\lambda x + (1-\lambda)y\|^2 \leq (1-\lambda)\|y\|^2 - \lambda\|y-x\|^2 \\ + \lambda \max\{\|y\|, 1\}\|x\|\beta(\|x\|) \\ + \lambda \max\{\|y\|, 1\}\|x-y\|\beta(\lambda\|x-y\|), \end{aligned}$$

where $0 \leq \lambda \leq 1, x, y \in X$.

Proof. By Theorem 6,

$$\begin{aligned} \|\lambda x + (1-\lambda)y\|^2 &= \|y + \lambda(x-y)\|^2 \\ &\leq \|y\|^2 + 2\lambda < x-y, J(y) > \\ &\quad + \lambda \max\{\|y\|, 1\}\|x-y\|\beta(\lambda\|x-y\|) \end{aligned}$$

On the other hand, by Theorem 6, $\|y-x\|^2 \leq \|y\|^2 - 2 < x, J(y) > + \max\{\|y\|, 1\}\|x\|\beta(\|x\|)$. Hence

$$\begin{aligned}
\|\lambda x + (1-\lambda)y\|^2 \leq\ & \|y\|^2 - 2\lambda\|y\|^2 - \lambda\|y-x\|^2 \\
& + \lambda\|y\|^2 + \lambda \max\{\|y\|, 1\}\|x\|\beta(\|x\|) \\
& + \lambda \max\{\|y\|, 1\}\|x-y\|\beta(\lambda\|x-y\|) \\
\leq\ & (1-\lambda)\|y\|^2 - \lambda\|y-x\|^2 \\
& + \lambda \max\{\|y\|, 1\}\|x\|\beta(\|x\|) \\
& + \lambda \max\{\|y\|, 1\}\|x-y\|\beta(\lambda\|x-y\|).
\end{aligned}$$

□

In [8] Rhoades prove that a unique fixed point for a quasi-contractive map exists. We obtain the conclusion that Mann iterative process converges to a unique fixed point of quasi-contractive map in uniformly smooth Banach space. The proof of Theorem 7 is based on the idea of Chidume and Osilike [1].

THEOREM 7. *Let X be uniformly smooth Banach space with β satisfying $\beta(t) \leq ct (c \geq 0)$. Let T be a quasi-contractive map from $B(X)$ to $B(X)$ with constant $k < \sqrt{\frac{1}{2c}}$. And we define $x_{n+1} = (1 - c_n)x_n + c_n T x_n$ where $0 \leq c_n \leq 1$, $\lim_{n\to\infty} c_n = 0, \sum_{n=1}^{\infty} c_n = \infty$. Then $\{x_n\}$ converges to the unique fixed point for T.*

Proof. Let x^* be a unique fixed point for T. Then

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =\ & \|(1-c_n)(x_n - x^*) + c_n(Tx_n - x^*)\|^2 \\
\leq\ & (1-c_n)\|x_n - x^*\|^2 - c_n\|x_n - Tx_n\|^2 \\
& + 2c_n\|Tx_n - x^*\|\beta(\|Tx_n - x^*\|) \\
& + 2c_n\|x_n - Tx_n\|\beta(c_n\|x_n - Tx_n\|) \\
\leq\ & (1-c_n)\|x_n - x^*\|^2 - c_n\|x_n - Tx_n\|^2 \\
& + 2c_n k^2(\|Tx_n - x_n\|^2 + \|x_n - x^*\|^2)c \\
& + 2c_n\beta(2c_n)\|x_n - Tx_n\|.
\end{aligned}$$

by the definition of quasi-contractive map T and the monotone map β. Hence

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le (1 - c_n(1 - 2k^2c))\ \|x_n - x^*\|^2 \\ + c_n(2k^2c - 1)\ \|Tx_n - x_n\|^2 \\ + 2c_n\beta(2c_n)\ \|Tx_n - x_n\|.\end{aligned}$$

Since $k < \frac{1}{\sqrt{2c}}$,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le (1 - c_n(1 - 2k^2c))\|x_n - x^*\|^2 \\ + 2c_n\beta(2c_n)\|Tx_n - x_n\|.\end{aligned}$$

By letting $\beta_n = \|x_n - x^*\|^2$ and Lemma 3, we have $\beta_n \to 0$. Therefore $x_n \to x^*$. □

Remark. In Theorem 7 if $c = 0$, then the theorem holds for any $k \in [0, 1)$. In Hilbert space we notice that β is constant zero.

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