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NON–CHARACTERISTIC CAUCHY PROBLEM FOR A PARABOLIC EQUATION

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§1.INTRODUCTION

The general parabolic differential operator of order $2p$ in one space variable is of the form

(1.1) $\mathcal{P} = \partial_x^{2p} + a_1(x,t)\partial_x^{2p-1} + a_2(x,t)\partial_x^{2p-2} + \cdots + a_{2p}(x,t) - c(x,t)\partial_t.$

We assume that all coefficients appeared in this paper are holomorphic in x and t. Consider a linear parabolic equation of second order in one space variable,

(1.2) $$\mathcal{L}[u] \equiv u_{xx} + a(x,t)u_x + b(x,t)u - c(x,t)u_t = F(x,t).$$

We observe that $t = 0$ is a characteristic.

It is well known that

(1.3) $$W(x,t:\xi,\tau) = \frac{1}{2\sqrt{\pi(\tau-t)}} exp\{-\frac{(\xi-x)^2}{4(\tau-t)}\}$$

is the fundamental solution of heat equation

$$\mathcal{M}[u] \equiv u_{xx} - u_t = 0,$$

which is the simplest form of (1.2). $t = \tau$ is the essential singularity of (1.3). We note that (1.3) is two-valued function. C.D.Hill[3] obtained a solution

$$\frac{\sqrt{\pi}(x-\xi)}{2(\tau-t)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(j+\frac{3}{2})} \{\frac{-(\xi-x)^2}{4(\tau-t)}\}^n$$

of the equation

(1.4) $$\mathcal{L}^t[v] \equiv v_{xx} - (a(x,t)v)_x + b(x,t)v + (c(x,t)v)_t = F(x,t),$$

under initial conditions

$$u|_{x=\xi} = 0, u_x|_{x=\xi} = \frac{-1}{t-\tau}$$

This paper is concerned with the non-characteristic Cauchy problem of parabolic differential equations whose initial condition has a pole.

§2. Construction of The Solution

We denote the adjoint equation $\mathcal{Q}v$ by

$$\begin{aligned}\mathcal{Q}v &= \partial^{2p}v - \partial^{2p-1}(a_1 v) + \partial^{2p-2}(a_2 v) - \cdots + a_{2p}v + (cv)_t \\ &\equiv Qv + (cv)_t.\end{aligned} \tag{2.1}$$

Now we state the following:

THEOREM. *There exist positive constants M, M_0 such that a solution of the initial value problem*

$$\begin{aligned}&\mathcal{Q}[v] = 0, \\ &\partial_x^{2p-1}v|_{x=\xi} = \frac{-1}{t-\tau}, \\ &\partial_x^k v|_{x=\xi} = 0, k = 0, 2, 3, \ldots, 2p-2\end{aligned} \tag{2.2}$$

is dominated by

$$M_0\frac{|x-\xi|^{2p-1}}{(2p-1)!|t-\tau|}exp\{M\frac{|x-\xi|^{2p}}{|t-\tau|}\}.$$

Proof. We take a Laurent expansion, as a solution, of the form:

$$S(x,t;\xi,\tau) = \sum_{j=0}^{\infty} S_j(x,t,\xi)\frac{j!}{(t-\tau)^{j+1}}. \tag{2.3}$$

By condition (2.2) we have

$$\begin{aligned}&S_j|_{x=\xi} = 0, j \geq 0, \\ &\partial_x^{2p-1}S_0|_{x=\xi} = -1, \partial_x^k S_0|_{x=\xi} = 0, \qquad 0 \leq k \leq 2p-2 \\ &\partial_x^k S_j|_{x=\xi} = 0, \qquad k = 0, 1, \cdots, 2p-2, j \geq 1.\end{aligned}$$

Since

$$\mathcal{Q}[S] = \frac{\mathcal{Q}[S_0]}{t-\tau} + \sum_{j=1}^{\infty}\{\mathcal{Q}[S_j] - cS_{j-1}\}\frac{j!}{(t-\tau)^{j+1}}.$$

$\mathcal{Q}[u] = 0$ is reduced to an infinite sequence of analytic Cauchy problem with data on the non-characteristic plane $x = \xi$

$$\begin{aligned} \mathcal{Q}[S_0] &= 0 \\ \mathcal{Q}[S_j] &= -cS_{j-1}, \qquad j = 1, 2, \cdots . \end{aligned} \tag{2.4}$$

They can be solved recursively by Cauchy-Kowalewski theorem. If all coefficient do not depend on t, S_j will be independent of t because of their constant Cauchy data. In that case we obtain a recursive system

$$\begin{aligned} Q[S_0] &= 0 \\ Q[S_j] &= -cS_{j-1}, \qquad j = 1, 2, \cdots . \end{aligned} \tag{2.5}$$

of ordinary differential equations. For simplicity we investigate the convergence issue only in the case in which the coefficients are independent of t. Then we note that $S_j(x,t,\xi) = S_j(x,\xi)$. Now we shall show that, for x and ξ in any compact interval $|x|, |\xi| \le h$, there exist constants M_0, M such that

$$|S_j(x,\xi)| \le M_0 M^j \frac{|x-\xi|^{2(j+1)p-1}}{(2(j+1)p-1)!} \tag{2.6}$$

On any compact interval $|x|, |\xi|, |\eta| \le h$ all coefficients are bounded. It follows by continuity and by Gronwall's lemma that there exists a constant M_0 such that

$$\begin{aligned} |\partial_x^{2p-1} S_0(x,\xi)| &\le M_0, \\ |S_0(x,\xi)| &\le M_0 \frac{|x-\xi|^{2p-1}}{(2p-1)!}. \end{aligned}$$

To estimate $S_j(x,\xi)$ we find the solution R of the following initial value problem:

$$\begin{aligned} &P[R] = 0, \\ &\partial_x^k R(t,t) = 0, k = 0, 1, \dots, 2p-2, \\ &\partial_x^{2p-1} R(t,t) = -1. \end{aligned} \tag{2.6}$$

Then we obtain

$$S_{j+1} = \int_\xi^x R(x,\eta)c(\eta)S_j(x,\xi)\,d\eta, j = 0, 1, \cdots .$$

It follows that by Gronwall's lemma and by induction there exist constants K, C such that

$$\begin{aligned} &|\partial_x^{2p-1}R(x,\eta)| \leq K, \\ &|R(x,\eta)| \leq K\frac{|x-\eta|^{2p-1}}{(2p-1)!}, \\ (2.7)\qquad &|c(x)| \leq C, \end{aligned}$$

for x, ξ, η contained in a compact interval. On the other hand we observe that

$$(2.8)\qquad |\int_\xi^x |x-\eta|\frac{|\eta-\xi|^{2l-1}}{(2l-1)!}\,d\eta| = \frac{|x-\xi|^{2l+1}}{(2l+1)!}.$$

Thus setting $KC = M$, we have

$$|S_1(x,\xi)| \leq M\frac{|x-\xi|^{4p-1}}{(4p-1)!}.$$

Thus it is not difficult to show that by induction

$$|S_j| \leq M^j\frac{|x-\xi|^{2(j+1)p-1}}{(2(j+1)p-1)!}.$$

Thus (2.5) follows. Since $(j!)^2 \leq (2j+1)!$ and

$$|S(x,t;\xi,\tau)| \leq M_0\frac{|x-\xi|^{2p-1}}{(2p-1)!|t-\tau|}\sum_{j=0}^\infty \frac{j!}{(2(j+1)p-1)!}\{\frac{M(x-\xi)^{2p}}{(t-\tau)}\}^j,$$

our theorem follows.

§3. Integral Representation

The identity

$$\int_D \{v\mathcal{P}u - u\mathcal{Q}v\}\,dx\,dt$$
$$= \int_{\partial D} \{\sum_{j=0}^{2p-2}\sum_{k=1}^{2p-j}(-\partial_x)^{k-1}(a_j v)\partial^{2p-j-k}u + a_{2p-1}uv\}\,dt + cuv\,dx$$

is valid in the complex domain as well as in the real domain. Suppose D is contained in a region where $\mathcal{P}[u] = F$ and $\mathcal{Q}[v] = 0$ and that its boundary $\partial D = \gamma_2 - \gamma_1$ has two components consisting of the cycles γ_2 and γ_1. Then

$$\oint_{\gamma_2} h[u,v] = \oint_{\gamma_1} h[u,v] + \int_D vF\,dx\,dt, \tag{3.1}$$

where

$$h[u,v] = \sum_{j=0}^{2p-2}\sum_{k=1}^{2p-j}\{(-\partial_x)^{k-1}(a_j v)\partial^{2p-j-k}u + a_{2p-1}uv\}\,dt + cuv\,dx.$$

If $F = 0$, the above integral is independent of path in the sense that its value is the same for any two cycles which are homologous in a region where $\mathcal{P}[u] \equiv \mathcal{Q}[v] \equiv 0$. Let the one dimensional cycle γ_0 be a loop about $t = \tau$ in the plane $x = \xi$, γ some other loop about $t = \tau$, and D the two dimensional chain composed of the lateral surface of a cylinder that wraps around $t = \tau$ and has γ_0 and γ as its two rims. In view of the initial conditions (2.2) we have

$$\oint_{\gamma_0} h[u,S] = \oint_{\gamma_0} \frac{u(\xi,t)}{t-\tau}\,dt = 2\pi i u(\xi,\tau).$$

Hence from (3.1) we obtain the representation

$$u(\xi,\tau) = \frac{1}{2\pi i}\oint_{\gamma} h[u,S] + \frac{1}{2\pi i}\int_D SF\,dx\,dt \tag{3.2}$$

for the solution to the Cauchy problem problem for $\mathcal{P}[u] = F$ with Cauchy data u and u_x given on γ.

References

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