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COHEN–MACAULAY PROPERTY OF GRADED RINGS ASSOCIATED TO EQUIMULTIPLE IDEALS

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1. Introduction.

The purpose of this paper is to extend C.Huneke and J.Sally's result ([2], Proposition 3.3) to the case of an equimultiple ideal.

Throughout this paper, all rings are assumed to be commutative with identity. By a local ring (R, m), we mean a Noetherian ring R which has a unique maximal ideal m. By $\dim(R)$ we always mean the Krull dimension of R. Let (R, m) be a local ring and I an ideal of R. An ideal J contained in I is called a reduction of I if $JI^n = I^{n+1}$ for some integer $n \geq 0$. A reduction J of I is called a minimal reduction of I if J is minimal with respect to being a reduction of I.
The reduction number of I with respect to J is defined by

$$r_J(I) = \min \{ n \geq 0 \mid JI^n = I^{n+1} \}.$$

The reduction number of I is defined by

$$r(I) = \min \{ r_J(I) \mid J \quad \text{is a minimal reduction of} \quad I \}.$$

The analytic spread of I, denoted by $l(I)$, is defined to be $\dim(R[It]/mR[It])$. In [5], it is shown that $ht(I) \leq l(I) \leq \dim(R)$. An ideal I is called equimultiple if $l(I) = ht(I)$. If R/m is an infinite field, then $l(I)$ is the least number of elements generating a reduction of I ([5]). In particular, all m-primary ideals are equimultiple. We will use notation $\lambda_R(M)$ (or simply $\lambda(M)$) to denote the length of M as an R-module, $\mu(I)$ to denote the number of elements in a minimal basis of an ideal I in R (i.e., $\mu(I) = \lambda(I/mI)$). As a general reference, we refer the reader to [4] for any unexplained notation or terminology.

2. Preliminaries.

In a d-dimensional local ring (R,m), for an m-primary ideal I, the Hilbert function $H_I(n) = \lambda(R/I^n)$ is a polynomial in n of degree d with leading coefficient $e(I)/d!$ for $n \geq 0$, i.e.,

$$e(I) = \lim_{n\to\infty} \frac{d!}{n^d}\lambda\big(R/I^n\big).$$

The integer $e(I)$ is called the multiplicity of I, and $e(R)$ means the multiplicity of the maximal ideal m of R.

THEOREM 2.1. ([1], (1)) If (R,m) is a Cohen-Macaulay local ring such that R/m is infinite, then $\mu(m) \leq e(R) + \dim(R) - 1$.
DEFINITION 2.2. A Cohen-Macaulay local ring (R,m) is said to have minimal multiplicity if $\mu(m) = e(R) + \dim(R) - 1$.
LEMMA 2.3. If a d-dimensional local ring (R,m) with infinite residue field satisfies the equation of minimal multiplicity, i.e.,$\mu(m) = e(R) + d - 1$, then R is Cohen-Macaulay if and only if for some minimal reduction J of m, $Jm = m^2$.
Proof : If $Jm = m^2$, then

$$\begin{aligned} e(J) &= e(R) \\ &= \mu(m) - d + 1 \\ &= \lambda(m/m^2) - \lambda(J/mJ) + 1 \\ &= \lambda(m/Jm) - \lambda(J/mJ) + 1 \\ &= \lambda(m/J) + \lambda(R/m) \\ &= \lambda(R/J). \end{aligned}$$

Hence R is Cohen-Macaulay by Theorem 17.11 of [4]. Conversely, if R is Cohen-Macaulay, then the assertion is clear by Theorem 1 in [6]. □
LEMMA 2.4. ([3], Lemma 3) Let $a_1, \cdots, a_n$ be elements of a ring A constituting an R-sequence in any order. Let J be an ideal generated by monomials in $a_2, \cdots, a_n$. Then $ba_1 \in J$ implies $b \in J$.

3. Main result.

THEOREM 3.1. Let (R, m) be a d-dimensional Cohen-Macaulay local ring with an infinite residue field R/m. Assume that I is an equimultiple ideal of R with $ht(I) = s > 0$ satisfying $I^2 = (a_1, \cdots, a_s)I$ for some minimal reduction $a_1, \cdots, a_s$ of I. Then $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal.

Proof : Let $r = d - s$. Since $\dim(R/I) = \dim(R) - ht(I)$, we can choose elements $b_1, \cdots, b_r \in m$ such that the images of the b_i in R/I are a system of parameters for R/I. Then $(a_1, \cdots, a_s, b_1, \cdots, b_r)$ is a system of parameters for R. By Corollary 2.7 in [8], the images $a_1^*, \cdots, a_s^*$ in I/I^2 form a regular sequence on $gr_I(R)$. Let $T = gr_I(R)/mgr_I(R)$ and $\dim(T) = l(I) = s$ by the definition of analytic spread.

Claim 1 : a_1' is a regular element on T.

We proceed by induction on $s = ht(I)$. We have two cases : (1) when $s = 1$, and (2) when $s > 1$.

Case(1) : $s = 1$. We have $I^2 = (a_1)I$. Suppose that a_1' is not a regular element on T. Then there exists a non-zero element $c' = c + mI^l$ in T such that $a_1'c' = 0$. Hence we have

$$a_1 c \in mI^{l+1} = m(a_1)I^l.$$

Since a_1 is a regular element on R, $c \in mI^l$, which is a contradiction.

Case(2) : $s > 1$. We have $I^2 = (a_1, \cdots, a_s)I$. Suppose that a_1' is not a regular element on T. Then there exists a non-zero element $c' = c + mI^l$ in T such that $a_1'c' = 0$. Hence we have

$$a_1 c \in mI^{l+1} = m(a_1, \cdots, a_s)I^l = mI(a_1, \cdots, a_s)^l.$$

Express $a_1 c$ as a homogeneous polynomial of degree l in $a_1, \cdots, a_s$ with coefficients in mI. Hence we have $a_1(c - h(a_1, \cdots, a_s)) = g(a_2, \cdots, a_s)$, where $h \in mI[a_1, \cdots, a_s]$ is a homogeneous polynomial of degree $l - 1$ in $a_1, \cdots, a_s$ with coefficients in mI and $g \in mI[a_2, \cdots, a_s]$ is a homogeneous polynomial of degree l in $a_2, \cdots, a_s$ with coefficients in mI, i.e.,

$$a_1(c - h(a_1, \cdots, a_s)) \in mI(a_2, \cdots, a_s)^l \subseteq (a_2, \cdots, a_s)^l.$$

By Lemma 2.4, we have $c - h(a_1, \cdots, a_s) \in (a_2, \cdots, a_s)^l$. Express $c - h(a_1, \cdots, a_s) = f(a_2, \cdots, a_s)$, where $f \in R[a_2, \cdots, a_s]$ is a homogeneous polynomial of degree l in $a_2, \cdots, a_s$ with coefficients in R. Moreover,

$$\begin{aligned} a_1 f(a_2, \cdots, a_s) &= a_1(c - h(a_1, \cdots, a_s)) \\ &\in mI(a_2, \cdots, a_s) \\ &\subseteq mI^{l+1}. \end{aligned}$$

By analytic independence, all the coefficients of f are in m. Hence we have

$$\begin{aligned} c &= h(a_1, \cdots, a_s) + f(a_2, \cdots, a_s) \\ &\in mI(a_1, \cdots, a_s)^{l+1} + mI^l \\ &\subseteq mI^l, \end{aligned}$$

which is a contradiction. This finishes the proof of claim 1. Pass to the Cohen-Macaulay local ring $R_1 = R/a_1R$. Then $I_1 = IR_1$ is an equimultiple ideal of $ht(I_1) = s - 1$ and $I_1(a_2, \cdots, a_s)R_1 = I_1^2$. The induction hypothesis applies to R_1, so $a_2', \cdots, a_s'$ form a regular sequence in $gr_{I_1}(R_1)/m_1 gr_{I_1}(R_1)$ and hence $gr_{I_1}(R_1)/m_1 gr_{I_1}(R_1)$ is Cohen-Macaulay, where $m_1 = mR_1$. But, since a_1^* is a regular element in $gr_I(R)$, we have the following isomorphism:

$$\frac{gr_{I_1}(R_1)}{m_1 gr_{I_1}(R_1)} \cong \frac{gr_I(R)/mgr_I(R)}{a_1'(gr_I(R)/mgr_I(R))}.$$

Hence $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay since a_1' is a regular element in T by claim 1. Let $N = I/mI \oplus I^2/mI^2 \oplus \cdots$ be the unique maximal homogeneous ideal of T. Hence T_N is a Cohen-Macaulay local ring of $\dim(T_N) = s$.

Claim 2 : T_N satisfies the equation of minimal multiplicity.
Since $a_1', \cdots, a_s' \in I/mI$, $(a_1', \cdots, a_s') \subseteq N$. Hence

$$\begin{aligned} (a_1', \cdots, a_s')N &= \frac{(a_1, \cdots, a_s)I}{mI^2} \oplus \frac{(a_1, \cdots, a_s)I^2}{mI^3} \oplus \cdots \\ &= I^2/mI^2 \oplus I^3/mI^3 \oplus \cdots \\ &= N^2. \end{aligned}$$

Therefore

$$\begin{aligned}
1+\mu(NT_N) &= \lambda(T_N/NT_N)+\lambda(NT_N/N^2T_N)\\
&= \lambda(T_N/N^2T_N)\\
&= \lambda(T_N/(a'_1,\cdots,a'_s)T_N)+\lambda((a'_1,\cdots,a'_s)T_N/N^2T_N)\\
&= \lambda(T_N/(a'_1,\cdots,a'_s)T_N)\\
&\quad +\lambda((a'_1,\cdots,a'_s)T_N/(a'_1,\cdots,a'_s)T_NNT_N)\\
&= e(T_N)+\mu((a'_1,\cdots,a'_s)T_N)\\
&= e(T_N)+s\\
&= e(T_N)+\dim(T_N). \qquad \square
\end{aligned}$$

COROLLARY 3.2. ([2], Proposition 3.3.) Let (R,m) be a d-dimensional Cohen-Macaulay local ring and I an m-primary ideal satisfying $I^2 = (a_1,\cdots,a_d)I$ for some minimal reduction $a_1,\cdots,a_d$ of I. Then $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal.

Proof : Any m-primary ideal in a local ring (R,m) is an equimultiple ideal. □

After this paper was completed, I learned that K.Shah proved independently the result in this title. But I used different technics to prove the main result.

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