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ON n–ROOT COMPACT HYPERGROUP

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0. Introduction.

The most important step on the way to the central limit theorem on the locally compact space is the embedding of an infinitely divisible measure into a continuous one-parameter convolution semigroup. The class of root compact groups enables us to describe the domain of validity for the theorem asserting the closedness of the infinitely divisible probability measures in the semigroup of all probability measures and for the fact that every infinitely divisible measures is submonogeneous embeddable. From these circumstances the class of root compact hypergroups provides a useful framework within which to study the embeddability on the semigroup of infinitely divisible probability measures. In [1], W. R. Bloom showed that this idea can be related to divisibility in the underlying hypergroup (see lemma 1.2 below). In this note we shall find conditions on the n-root compact hypergroup.

I. Preliminaries.

Let G be a locally compact space. By $\mathbf{M}^1(G)$ we denote the space of probability measures on G furnished with the weak topology $\mathcal{T}_w$. The Dirac measure in a point $x \in G$ will be abbreviated by ε_x. Then G is said to be a ***weak hypergroup*** in the sense of Jewett [4] if the following conditions are satisfied :

(H1) There exists a mapping $(x, y) \mapsto \varepsilon_x * \varepsilon_y$ from $G \times G$ into $\mathbf{M}^1(G)$ which is continuous and satisfies the equality

(1.0.1) $$(\varepsilon_x * \varepsilon_y) * \varepsilon_z = \varepsilon_x * (\varepsilon_y * \varepsilon_z) \qquad \text{for all } x, y, z \text{ in } G.$$

(H2) For any x, y in G $\operatorname{supp}(\varepsilon_x * \varepsilon_y)$, the support of $\varepsilon_x * \varepsilon_y$, is compact.

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(H3) There exists a homeomorphism $x \mapsto x^{-1}$ from G into G such that for all x, y in G, the equalities $x = (x^{-1})^{-1}$ and $(\varepsilon_x * \varepsilon_y)^{-1} = \varepsilon_{y^{-1}} * \varepsilon_{x^{-1}}$ hold.

(H4) There exists an element e in G such that $\varepsilon_e * \varepsilon_x = \varepsilon_x * \varepsilon_e = \varepsilon_x$ for all $x \in G$.

The mapping "$*$" introduced in (H1) is called the convolution in G ; it can be extended to a convolution in the set $\mathbf{M}^b(G)$ of all bounded measures on G. The homeomorphism $x \mapsto x^{-1}$ introduced in (H3) is called the involution of G ; it can also be extended to $\mathbf{M}^b(G)$. Naturally the element "e" defined in (H4) is said to be the unit element of G. A weak hypergroup G is called a ***hypergroup*** if the following additional conditions are satisfied :

(H5) For any x, y in G, $e \in \text{supp}(\varepsilon_x * \varepsilon_{y^{-1}})$ iff $x = y$.

(H6) The mapping $(x, y) \mapsto \text{supp}(\varepsilon_x * \varepsilon_y)$ from $G \times G$ into $\mathcal{R}(G)$, where $\mathcal{R}(G)$ is the space of all nonempty compact subsets of G, is continuous.

A hypergroup G is said to be a ***commutative*** if $\varepsilon_x * \varepsilon_y = \varepsilon_y * \varepsilon_x$ for any x, y in G. Given $f \in \mathcal{C}^b(G)$ and x, y in G, we write

$$f(x * y) = \int_G f(z)\, \varepsilon_x * \varepsilon_y(dz) \tag{1.0.2}$$

where $x * y$ is considered as the set $\text{supp}(\varepsilon_x * \varepsilon_y)$. If A and B are subsets of G then the set $A * B$ is denoted by

$$A * B := \bigcup_{\substack{x \in A \\ y \in B}} \text{supp}(\varepsilon_x * \varepsilon_y) \tag{1.0.3}$$

If H is a compact subhypergroup of the hypergroup G and $x \in G$, then the sets $H * \{x\} * H := HxH$ form a decomposition of G, and $G//H = \{HxH : x \in G\}$ has a natural convolution. This set $G//H$ will be called ***double cosets*** of H. Then with the normalized Haar measure ω_H on H and following operators :

$$\varepsilon_{HxH} * \varepsilon_{HyH} := \int_H \varepsilon_{HxtyH}\, \omega_H(dt) \qquad \text{for any} \quad x, y \in G$$

and

$$(HxH)^{-1} := Hx^{-1}H,$$

each double coset $G//H$ is compact hypergroup furnished with the quotient topology and the identity element is $H = HeH$ (cf: [4]; Theorem 8.2B). For $x, y \in G$ write $x \sim y$ if there exist $z_1, z_2, \cdots, z_n \in G$ such that $y \in Z_n * \{x\} * Z_n^{-1}$, where $Z_n = \{z_1\} * \{z_2\} * \cdots * \{z_n\}$. The relation $\sim$ is a equivalence relation on G (cf : [1]; Lemma 3.1). We write

$$C_x = \{ y \in G : y \sim x \}$$

for the **conjugacy class** of G containing x. Then we can convert the set C_x into following :

$$C_x = \bigcup \{ Z_n * \{x\} * Z_n^{-1} : Z_n = \{z_1\} * \{z_2\} * \cdots \{z_n\}, \quad z_i \in G, i = 1, 2, \cdots, n, n \in \mathbb{N} \}$$

A hypergroup G is called **class compact** if all of its conjugacy classes are relatively compact, in other words, the closure of C_x is compact for any $x \in G$.

DEFINITION 1.1. *Let $n \in \mathbb{N}$. A hypergroup G is called **n-root compact** (written $G \in \mathcal{R}_n$) if the following condition holds : For every compact $C \subset G$ there exists compact $C_n \subset G$ such that all finite sets $\{x_1, x_2, \cdots, x_n\}$ in G with $x_n = e$ satisfying*

(1.1.1) $$\{x_i\} * C * \{x_j\} * C \bigcap \{x_{i+j}\} * C \neq \phi$$

for $i + j \leq n$ are contained in C_n.

Write $\mathcal{R} := \bigcap_{n \geq 1} \mathcal{R}_n$ for the class of all root compact hypergroups. G is said to be ***strongly n-root compact*** if G is n-root compact with a subset $C_n = C_0$ independent of n. For each $K \subset \mathbf{M}^1(G)$ and every compact $C \subset G$, write

(1.1.2) $$\mathcal{R}(n, K) = \{ \mu \in \mathbf{M}^1(G) : \mu^n \in K \},$$

(1.1.3) $$\mathcal{R}(n, \mu) = \{ \nu \in \mathbf{M}^1(G) : \nu^n = \mu \},$$

(1.1.4) $$C^{1/n} = \{ x \in G : \{x\}^n \subset C \}.$$

LEMMA 1.2. *([1]; Theorem 4.4) Let G be a hypergroup and consider the following conditions :*

(i) $G \in \mathcal{R}_n$;

(ii) $\mathcal{R}(n, K)$ *is relatively compact for each relative compact* $K \subset \mathbf{M}^1(G)$

(iii) $C^{1/n}$ *is compact for every compact* $C \subset G$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If, in addition, G is class compact and has a compact invariant neighborhood of e, then* (iii) $\Rightarrow$ (i) *also holds.*

II. Main result

We now meet the conditions satisfying the n-root compactness of a hypergroup G. Our following statements are the extension to a hypergroup for the n-root compact locally compact groups.

THEOREM 2.1. *Let G be a hypergroup, H a compact normal subhypergroup of G, the mapping $\pi : G \to G//H$ given by $\pi(x) = HxH$ a continuous epimorphism from G onto $G//H$. Then $G \in \mathcal{R}_n$ iff $G//H \in \mathcal{R}_n$.*

Proof. Suppose that $G \in \mathcal{R}_n$ and $G' = G//H$. Choose a compact subset D of G' and write $C = \pi^{-1}(D)$. Then C is a compact subset of G ([4]; Theorem 13.1.c). For any finite set $\{x_1, x_2, \cdots, x_n\}$ in G with $x_n = e$, put

(2.1.1)

$$\{Hx_iH\} * D * \{Hx_jH\} * D \bigcap \{Hx_{i+j}H\} * D \neq \phi \quad \text{for} \quad i + j \leq n.$$

'We shall prove that, for all finite subset $\{Hx_1H, Hx_2H, \cdots, Hx_nH\} \subset G'$ with $Hx_nH = H$, there exists a compact subset D_n of G' satisfying the equation (2.1.1). But then from (2.1.1),

$$\begin{aligned}
\phi &\neq \{\pi(x_i)\} * \pi(C) * \{\pi(x_j)\} * \pi(C) \bigcap \{\pi(x_{i+j})\} * \pi(C) \\
&= \pi[\{x_i\} * H * C] * \pi[\{x_j\} * H * C] \bigcap \pi[\{x_{i+j}\} * H * C] \\
&= \pi[\{x_i\} * H * C * H * \{x_j\} * H * C] \bigcap \pi[\{x_{i+j}\} * H * C] \\
&= H * \{x_i\} * H * C * H * \{x_j\} * H * C * H \bigcap H * \{x_{i+j}\} * H * C * H
\end{aligned}$$

by the normality of H

$$= \{x_i\} * H^2 * C * H * \{x_j\} * H * C * H \bigcap \{x_{i+j}\} * H^2 * C * H.$$

Since H is a subhypergroup of G, it follows that

$$(2.1.2)\quad \phi \neq \{x_i\}*(H*C*H)*\{x_j\}*(H*C*H) \bigcap \{x_{i+j}\}*(H*C*H).$$

Moreover since H and C are compact, so is $H * C * H$, there exists a compact subset C_n of G containing $\{x_1, x_2, \cdots, x_n\}$ with $x_n = e$ that satisfies the equation (2.1.2). Now define $D_n := \pi(C_n)$, then D_n is compact and contains a finite subset $\{\pi(x_1), \pi(x_2), \cdots, \pi(x_n)\}$ of G', and hence satisfies the equation (2.1.1).

Suppose that $G' \in \mathcal{R}_n$ and C is a compact subset of G. Then $D = \pi(C)$ is compact in G', so that for any finite subset $\{Hx_1H, Hx_2H, \cdots, Hx_nH\}$ with $Hx_nH = H$, there is a compact subset D_n of G' satisfying

(2.1.3)

$$\{Hx_iH\} * D * \{Hx_jH\} * D \bigcap \{Hx_{i+j}H\} * D \neq \phi \quad \text{for} \quad i+j \leq n.$$

Let, now, $C_n = \pi^{-1}(D_n)$, then C_n is compact (by [4],Theorem 13.1.c), and let any finite subset $\{x_1, x_2, \cdots, x_n\} \subset G$ satisfy

$$(2.1.4)\qquad \{x_i\} * C * \{x_j\} * C \bigcap \{x_{i+j}\} * C \neq \phi \quad \text{for} \quad i+j \leq n,$$

where $x_n = e$. Then we have

$$(2.1.5)\qquad \begin{aligned} &\phi \neq \pi[\{x_i\} * C * \{x_j\} * C \bigcap \{x_{i+j}\} * C] \\ &\subset \pi[\{x_i\} * C * \{x_j\} * C] \bigcap \pi[\{x_{i+j}\} * C] \\ &= \{\pi(x_i)\} * D * \{\pi(x_j)\} * D \bigcap \{\pi(x_{i+j})\} * D \end{aligned}$$

and hence, $\{\pi(x_1), \pi(x_2), \cdots, \pi(x_n)\} \subset D_n$, so $\{x_1, x_2, \cdots, x_n\} \subset C_n$. Moreover from the definition of π, $\pi(x_n) = Hx_nH = H$, and therefore $x_n = e$. ∎

THEOREM 2.2. *Let G be a class compact hypergroup with a compact invariant neighborhood of e, $\mathcal{H}$ an ascending system of normal* *subhypergroups in $\mathcal{R}_n$ with $G = \cup\{H : H \in \mathcal{H}\}$ such that for every $H \in \mathcal{H}$ there exists only a finite number of elements in $G//H$ whose order divides n. Then $G \in \mathcal{R}_n$.*

Proof. Without loss of generality, let C be a G-invariant compact subset of G and assume that $e \in C$, $C = C^{-1}$ where $C^{-1} = \{y : \varepsilon_x * \varepsilon_y = \varepsilon_e$ for all $x \in C\}$. Put $C^{1/n} = \{x \in G : \{x\}^n \subset C\}$ for $n \geq 1$. It is sufficient to show that $C^{1/n}$ is compact in G by lemma 1.2. Choose $H \in \mathcal{H}$ such that $C \subset H$, and denote the natural projection $\pi : G \to G//H$, given by $\pi(x) = HxH$. Then, by assumption, there exists a finite subset $\{x_1, x_2, \cdots, x_r\}$ in G such that $\pi(x_1), \pi(x_2), \cdots \pi(x_r) \in G//H$ are elements whose order divides n. Let $x \in C^{1/n}$. Then $\{x\}^n \subset C \subset H$, and hence $\{\pi(x)\}^n = Hx^nH = \pi(e)$. Thus there is a $j \in \{1, 2, \cdots, r\}$ with $x \in Hx_jH$, and by the normality of H, $x \in x_jH^2$ or $x \in x_jH$. Hence there are an $h \in H$ and an invariant compact subset D of G such that $h = xx_j^{-1}$ and $\{x_1^{-1}, x_2^{-1}, \cdots, x_r^{-1}\} \subset D$. Then $h \in \{x\}D \subset CD$ and $\{h\}^n \subset CD^n$, and since CD^n is compact there exists an $H' \in \mathcal{H}$ with $H \subset H'$, $CD^n \subset H'$. The set $D^{1/n} := \{h \in H' : \{h\}^n \subset CD^n\}$ is compact since $H' \in \mathcal{R}_n$. But $h \in H \subset H'$ and $\{h\}^n \subset CD^n$, hence $h \in D^{1/n}$. So $x = hx_j \in D^{1/n} * \{x_j\}$, or $C^{1/n} \subset \bigcup_{j=1}^{r} D^{1/n} * \{x_j\}$. Therefore $C^{1/n}$ is compact. □

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