Kangweon–Kyungki Math. Jour. 1 (1993), pp. 47–53

AN COMPLETION OF SPACE OF FUZZY RANDOM VARIABLES

YUN KYONG KIM AND BYUNG MOON KIL

1. Introduction

Fuzzy random variables generalize random sets which is an extension of random variables and random vectors. Kwakernaak[5] introduced the notion of a fuzzy random variable as a function $F : \Omega \to \overline{\mathcal{F}}(R)$ subject to certain measurability conditions,where(Ω, Σ, P) is a probability space, and $\overline{\mathcal{F}}(R)$ denotes all piecewise continuous functions $u : R \to [0, 1]$. Puri and Ralescu[7] defined a fuzzy random variable by a function $X : \Omega \to \mathcal{F}_o(R^n)$ subject to certain measurability requirements, where $\mathcal{F}_o(R^n)$ denotes all functions $u : R^n \to [0, 1]$ such that $\{x \in R^n : u(x) \geq \alpha\}$ is nonempty and compact for each $0 < \alpha \leq 1$, and proved an completion of $\mathcal{F}_o(R^n)$ with respect to an appropriate metric. Stojakovic[9] defined the notion of a fuzzy random variable slightly different than that in [5] and [7], and proved that the space of integrably bounded fuzzy random variables is complete with repect to a new metric.

In this paper, we adopt the notoin of a fuzzy random variable in Puri and Ralescu[7], and the space of integrably bounded fuzzy random variables is complete with respect to the metric introduced in Stojakovic[9].

2. Preliminaries

Throughout this paper, let (Ω, Σ, P) be a probability space and Λ a real separable Banach space with norm $\|\ \ \|$. Let $\mathcal{K}(\Lambda)$ denotes the family of all nonempty, compact subsets of Λ and $\mathcal{K}_c(\Lambda)$ the family of all nonempty, compact, and convex subsets of Λ. A linear structure in $\mathcal{K}(\Lambda)$ is defined via the operations

$$A + B = \{a + b : a \in A, b \in B\}$$
$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in \mathcal{K}(\Lambda), \lambda \in R$. However, note that $\mathcal{K}(\Lambda)$ is not a vector space since $A + (-A) \neq \{0\}$.

The topology in $\mathcal{K}(\Lambda)$ is introduced through the Hausdorff metric

$$H(A,B) = max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$$

We denote the Hausdorff semimetric by

$$h(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|$$

.

It is well-known that $\mathcal{K}(\Lambda)$ is a complete and separable metric space, and that $\mathcal{K}_c(\Lambda)$ is a closed subspace.
Let $L(\Omega, \Sigma, P, \Lambda) = L$ denotes the Banach space of (equivalence classes of) measurable functions $f : \Omega \to \Lambda$ such that the norm $\|f\|_1 = \int_\Omega \|f(\omega)\|\, dP$ is finite.
A random set is defined as a Borel measurable function $F : \Omega \to \mathcal{K}(\Lambda)$, and a measurable function $f : \Omega \to \Lambda$ is called a measurable selection of F if $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$. For a random set F, define the set $S_F = \{f \in L : f(\omega) \in F(\omega) a.e.\}$ then , S_F is a closed subset of L. If $F : \Omega \to \mathcal{K}(\Lambda)$ is a random set, the expectation of F is defined by $\int_\Omega F\, dP = \{\int_\Omega f\, dP : f \in S_F\}$ where $\int_\Omega f\, dP$ is the Bochner-integral. A random set $F : \Omega \to \mathcal{K}(\Lambda)$ is called integrably bounded if there exists integrable function $g : \Omega \to R$ such that $\sup_{x\in F(\omega)} \|x\| \leq g(\omega)$ for all $\omega \in \Omega$. Let $\mathcal{L}(\Omega, \Sigma, P, \Lambda) = \mathcal{L}$ denote the space of all integably bounded random sets, where $F, G \in \mathcal{L}$ are considered to be identical if $F(\omega) = G(\omega)$ a.s.. For $F, G \in \mathcal{L}$, we define

$$\Delta(F,G) = \int_\Omega H\{F(\omega), G(\omega)\}\, dP$$

$$\delta(F,G) = \int_\Omega h\{F(\omega), G(\omega)\}\, dP$$

Then Δ is a metric and δ is a semimertic on $\mathcal{L}$. If we define

$$\mathcal{L}_c(\Omega, \Sigma, P, \Lambda) = \mathcal{L}_c = \{F \in \mathcal{L} : F(\omega) \in \mathcal{K}_c(\Lambda) a.s.\}$$

then $\mathcal{L}$ is a complete metric space with respect to the metric Δ and $\mathcal{L}_c$ is a closed subspace of $\mathcal{L}$ [3].

3. Fuzzy random variables

A Fuzzy set in Λ is a function $u : \Lambda \to [0,1]$. Denote by $L_\alpha u = \{x \in \Lambda | u(x) \geq \alpha\}$ for $0 \leq \alpha \leq 1$, the α-level set of u. An extension of $\mathcal{K}(\Lambda)$ is obtained by defining the space $\mathcal{F}(\Lambda)$ of all fuzzy sets $u : \Lambda \to [0,1]$ with the properties

(a) u is upper semicontinuous

(b) supp u is compact

(c) $\{x \in \Lambda | u(x) = 1\} \neq \emptyset$

The space $\mathcal{F}_c(\Lambda)$ denotes the family of all fuzzy sets in $\mathcal{F}(\Lambda)$ which are also fuzzy convex. It is clear that $A \in \mathcal{K}(\Lambda)$ implies $\chi_A \in \mathcal{F}(\Lambda)$, while $A \in \mathcal{F}_c(\Lambda)$ implies $\chi_A \in \mathcal{F}_c(\Lambda)$, where χ_A is the indicator function of A.

A linear structure in $\mathcal{F}(\Lambda)$ is defined by the operation

$$(u+v)(x) = \sup_{y+z=x} min[u(y), v(z)]$$

$$(\lambda u)(x) = \begin{cases} u(x/\lambda), \text{if } \lambda \neq 0 \\ \chi_{\{0\}}(x), \text{if } \lambda = 0. \end{cases}$$

where $u, v \in \mathcal{F}(\Lambda)$ and $\lambda \in R$.

A fuzzy random variable is defined as a function $X : \Omega \to \mathcal{F}(\Lambda)$ such that $L_\alpha X : \Omega \to \mathcal{K}(\mathcal{X}), L_\alpha X(\omega) = \{x \in \Lambda : X(\omega)(x) \geq \alpha\}$ is a random set for all $\alpha \in [0,1]$. A fuzzy random variable X is called integrably bounded if $L_\alpha X$ is integrably bounded for all $\alpha \in [0,1]$. Let $\Phi(\Omega, \Sigma, P, \Lambda) = \Phi$ be the set of all integrably bounded fuzzy random variables. With Φ_c we denote the set of all fuzzy random variables $X \in \Phi$ such that $L_\alpha X \in \mathcal{L}_c$ for all $\alpha \in (0,1]$.

4. Main Result

For $X, Y \in \Phi$, we define $D(X,Y) = \sup_{\alpha \geq 0} \Delta(L_\alpha X, L_\alpha Y)$. Two fuzzy random variables $X, Y \in \Phi$ are considered to be identical if $L_\alpha X = L_\alpha Y$ a.s. for all $\alpha \in [0,1]$. It is obvious that D is a metric in Φ and if

F,G are integrably bounded random set then $D(F,G) = \Delta(F,G)$.
To prove the main result, we need the following lemma.

Lemma 4.1. Let $\{F_\alpha : \alpha \in [0,1]\}$ be a family of random sets such that

(a) $F_0(\omega) = \Lambda$ for all $\omega \in \Omega$

(b) $\alpha \leq \beta$ implies $F_\beta \subseteq F_\alpha a.s.$

(c) $\alpha_1 \leq \alpha_2 \leq \dots, \lim \alpha_n = \alpha$ implies $F_\alpha = \bigcap_{n=1}^{\infty} F_{\alpha_n}$ a.s.

Then the fuzzy random variable $X : \Omega \to \mathcal{F}(\mathcal{X})$ defined by $X(\omega)(x) = \sup\{\alpha \in [0,1] : x \in F_\alpha(\omega)\}$ has the property that $L_\alpha X = F_\alpha$ for every $\alpha \in [0,1]$.

Proof. It follows immediately from an application of lemma 1 [9].

Theorem 4.2. Φ is a complete metric space with repect to the metric D, and Φ_c is a closed subspace of Φ.

Proof. Let $\{X_n, n \geq 1\}$ be a Cachy sequence in Φ. Consider a fixed $\alpha > 0$. Then $\{L_\alpha(X_n), n \geq 1\}$ is a Cachy sequence in $\mathcal{L}$. Since $\mathcal{L}$ is complete with respect to Δ, it follows that

$$L_\alpha(X_n) \xrightarrow{\Delta} F_\alpha \in \mathcal{L}.$$

Actually, it is easy to see that $L_\alpha(X_n) \xrightarrow{\Delta} F_\alpha$ uniformly in $\alpha \in [0,1]$. Consider now the family $\{F_\alpha : \alpha \in [0,1]\}$, where $F_0(\omega) = \Lambda$ for all $\omega \in \Omega$.

Let $\varepsilon > 0$ and $\alpha \leq \beta$. Then

$$\delta(F_\beta, F_\alpha) \leq \delta(F_\beta, L_\beta(X_n) + \delta(L_\beta(X_n), L_\alpha(X_n)) + \delta(L_\alpha(X_n), F_\alpha)$$

Since $L_\beta(X_n) \subset L_\alpha(X_n)$, it follows that $\delta(L_\beta(X_n), L_\alpha(X_n)) = 0$.
Thus, $\delta(F_\beta, F_\alpha) \leq \delta(F_\beta, L_\beta(X_n)) + \delta(L_\alpha(X_n), F_\alpha) < \varepsilon$ if n is large enough. Hence $\delta(F_\beta, F_\alpha) = 0$ and since $F_\beta(\omega), F_\alpha(\omega)$ are closed, we have $F_\beta(\omega) \subseteq F_\alpha(\omega) a.s.$
Now take $\alpha > 0, \alpha_n \uparrow \alpha$. We have to show that

$$F_\alpha = \bigcap_{n=1}^{\infty} F_{\alpha_n} a.s.$$

It is clear that $F_\alpha \subseteq \bigcap_{n=1}^{\infty} F_{\alpha_n}$ a. s.
Using again the semimetric δ , we get for fixed j,

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, F_\alpha) \leq \delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) + \delta(\bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j), L_\alpha(X_j)) + \delta(L_\alpha(X_j), F_\alpha)$$

But $\delta(\bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j), L_\alpha(X_j)) = 0$. Conseqently, for every $\varepsilon > 0$, there exits N_ε such that for $j \geq N_\varepsilon$

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, F_\alpha) \leq \varepsilon + \delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j))$$

Now, for any $k \geq 1$,

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) \leq \delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, F_{\alpha_k}) + \delta(F_{\alpha_k}, L_{\alpha_k}(X_j)) + \delta(L_{\alpha_k}(X_j), \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)).$$

Since $\bigcap_{n=1}^{\infty} F_{\alpha_n} \subseteq F_{\alpha_k}$, we obtain

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) \leq \delta(M_{\alpha_k}, L_{\alpha_k}(X_j)) + \delta(L_{\alpha_k}(X_j), \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j))$$

Now $\delta(F_{\alpha_k}, L_{\alpha_k}(X_j)) < \varepsilon$ for $j \geq N_0$. Note that N_0 does not depend on k since the convergence $L_\alpha(X_j) \to F_\alpha$ is uniform. On the other hand, since $\{L_{\alpha_n}(X_j)\}$ decrease to $\bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)$, it follows that $\delta(L_{\alpha_m}(X_j), \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) < \varepsilon$ for some m (depending on j). Thus, if j is large,

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, \bigcap_{n=1}^{\infty} L_{\alpha_n}(X_j)) < 2\varepsilon$$

Finally by taking j large enough, we obtain

$$\delta(\bigcap_{n=1}^{\infty} F_{\alpha_n}, F_\alpha) \leq 3\varepsilon$$

i.e.,

$$\bigcap_{n=1}^{\infty} F_{\alpha_n} \subseteq F_\alpha a.s.$$

Hence we obtain $\bigcap_{n=1}^{\infty} F_{\alpha_n} = F_\alpha a.s.$ Thus lemma4.1 is applicable and there exists $X \in \Phi$ with $L_\alpha(X) = F_\alpha$ for every $\alpha \in [0,1]$. It remains to show that $X_n \to X$ with respect to D. Let $\varepsilon > 0$. Then since $\{X_n\}$ is Cauchy, there exists N_ε such that $n, m > N_\varepsilon$ implies $D(X_n, X_m) < \varepsilon$. Let $n > N_\varepsilon$ be fixed. Then

$$\begin{aligned} D(L_\alpha(X_n), L_\alpha(X)) &= \lim_{m\to\infty} D(L_\alpha(X_n), L_\alpha(X_m)) \\ &\leq \overline{\lim_{m\to\infty}} \sup_{\alpha>0} D(L_\alpha(X_n), L_\alpha(X_m)) \\ &= \overline{\lim} D(X_n, X_m) < \varepsilon \end{aligned}$$

Thus,

$$D(X_n, X) = \sup_{\alpha>0} D(L_\alpha(X_n), L_\alpha(X)) \leq \varepsilon$$

for $n > N_\varepsilon$.
This completes the proof of the first statement and the second statement is trivial.

References

1. R. J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl. **12** (1965), 1–12.
2. G. Debreu, *Integration of corresponsence,*, Proceedings, Fifth Berkeley Symp. Math. Statist. Prob. **Vol.2** (1966), 351–372.
3. F. Hiai and H. Umegaki, *Integrals, Conditional expectations, and martingales of multivalued functions*, J. Multivariate Anal. **7** (1977), 149–182.
4. E. P. Klement, M. L. Puri and D. A. Ralescu, *Limit theorems for fuzzy random variables*, Proc. Roy. Soc. London Ser. A **407** (1986), 171–182.
5. H. Kwakernaak, *Fuzzy random variables, Definition and theorems*, Inform. Sci. **15** (1978), 1–29.
6. C. V. Negoita and D. A. Ralescu, *Applications of Fuzzy sets to System Analysis*, Wiley, New York, 1975.
7. M. L. Puri and D. A. Ralescu, *Fuzzy random variables*, J. Math. Anal. Appl. **114** (1986), 402–422.

8. M. L. Puri and D. A. Ralescu, *Convergence theorem for fuzzy martingales*, J. Math. Anal. Appl. **160** (1991), 107–122.
9. M. Stojakovic, *Fuzzy conditional expectation*, Fuzzy sets and Systems **52** (1992), 53–60.
10. L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338–353.

Yun Kyong Kim
Dept. of Mathematics,
Dong-shin University,
Naju 520-714, Korea

Byung Moon Kil
Department of Mathematics,
Sung–Hwa University,
Chunan 330-150, Korea