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A NOTE ON EXTENSION OF STABILITY OF HARMONIC MAPS

DONG GWEON CHUNG

1. Introduction.

It is well known that many interesting subjects in geometry, for instance geodesics, Einstein metrics, minimal submanifolds and Yang-Mills fields appear as the type of variational problem. The theory of harmonic maps is the natural extension of the notions of geodesic and minimal submanifold, and has recently developed very much as we look excellent expository reports [1, 2] of Eells and Lemaire. Let ϕ be a smooth map of a compact Riemannian manifold M into another Riemannian manifold N. The energy functional $E(\phi)$ for ϕ is defined by

$$E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 \, dv_M,$$

where $d\phi$ denotes the differential of ϕ and dv_M is the volume element of M. A smooth map ϕ of M into N is called a *harmonic map* if ϕ is a critical point of the energy functional E, i.e., the tension field of ϕ vanishes identically. Harmonic maps apper in many different contexts according to what M and N are.

In this note we will extend the stability theory of harmonic maps to the case of pluriharmonic maps. In order to do this, in section 2 we present the Morse-Schoenberg theorem and deal with some earlier results about the notions of the index and nullity of harmonic maps, moreover look into what kind of harmonic maps are stable. In section 3 we investigate some conditions of the stability of pluriharmonic maps. Finally we induce the following theorem 11 from several lemmas.

THEOREM 11. *Let (M, g) and (N, h) be compact Riemannian manifolds and let a smooth map $\phi : M \to N$ be a non-constant isometric totally geodesic immersion. Then for a large number p, ϕ is p-stable if and only if ϕ is a minimal immersion and stable.*

2. Stability of harmonic maps.

First of all we treat the variation formulae for harmonic maps. Let (M,g) and (N,h) be complete Riemannian manifolds of dimension m and n respectively. Let $C^\infty(M,N)$ be the set of all smooth maps of M into N and $C^\infty(M)$ the set of all real-valued functions on M. Let $\chi(M)$ be the set of all smooth vector fields on M. For the map $\phi \in C^\infty(M,N)$ we define the energy density function $e(\phi) \in C^\infty(M)$ by

$$\begin{aligned} e(\phi)(x) &= \frac{1}{2}h(\phi_*,\phi_*)(x) \\ &= \frac{1}{2}\sum_{i=1}^{m} h(\phi_* e_i, \phi_* e_i)(\phi(x)), x \in M, \end{aligned} \tag{2-1}$$

where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame filed on M, $h(\phi_*,\phi_*)(x)$ denotes the Hilbert-Schmidt norm of the differential ϕ_* : $(T_xM, g_x) \to (T_{\phi(x)}N, h_{\phi(x)})$ at the point x. For a relatively compact domain Ω in M, the energy $E(\Omega,\phi)$ of ϕ on Ω is defined by

$$E(\Omega,\phi) = \frac{1}{2}\int_\Omega \sum_{i=1}^{m} h(\phi_* e_i, \phi_* e_i) * 1, \tag{2-2}$$

where $*1$ is the volume element of M. We denote $E(\phi) = E(M,\phi)$ when defined (in fact, if M is compact), and call as it the energy of ϕ. Let $\phi_t \in C^\infty(M,N)$, $|t| < \epsilon$, be a one-parameter family of maps from M into N with $\phi_0 = \phi$. Then since for all $x \in M$,

$$V_x = \left.\frac{d}{dt}\right|_{t=0} \phi_t(x) \in T_{\phi(x)}N, \tag{2-3}$$

a variation vector field V along ϕ with $V \equiv 0$ on $\partial\Omega$ is given. Let $\phi^{-1}TN$ be the bundle induced by ϕ over M from the tangent bundle TN of N, and $\Gamma(\phi^{-1}TN)$ the space of all sections V of $\phi^{-1}TN$. That is, $V \in \Gamma(\phi^{-1}TN)$ means that V is a map of M into $\phi^{-1}TN$ such that $V_x \in T_{\phi(x)}N$ for all $x \in M$. Let ∇ and ${}^N\nabla$ be the Levi-Civita connections of (M, g) and (N, h) respectively. For a tangent vector X in M, the induced connection $\tilde{\nabla}$ on $\phi^{-1}TN$ is defined by

$$\nabla_X V = {}^N\nabla_{\phi_* X} V, \qquad V \in \Gamma(\phi^{-1}TN). \tag{2-4}$$

Furthermore we define the tension filed $\tau(\phi) \in \Gamma(\phi^{-1}TN)$ of ϕ by

$$\tau(\phi) = \sum_{i=1}^{n} (\tilde{\nabla}_{e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i). \tag{2-5}$$

Here we get the first variation formula as follows;

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = - \int_M h(V, \tau(\phi)) * 1. \tag{2-6}$$

The first variation formula (2-6) shows that harmonic maps are critical points of the energy functional $E(\phi)$, and that any differential map ϕ with vanished tension field $\tau(\phi)$ is a harmonic map. For any given harmonic map $\phi \in C^\infty(M, N)$, if $V \in \Gamma(\phi^{-1}TN)$ has compact support, then the second variation formula of the energy E is given by

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\phi_t) = - \int_M h(V, J_\phi V) * 1, \tag{2-7}$$

where $J_\phi : \Gamma(\phi^{-1}TN) \to \Gamma(\phi^{-1}TN)$, the Jacobi operater of ϕ, is a second order elliptic differential operator given by

$$J_\phi V = - \sum_{i=1}^{m} \{\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} V - \tilde{\nabla}_{\nabla_{e_i} e_i} V\} - \sum_{i=1}^{m} {}^N R(\phi_* e_i, V) \phi_* e_i, \tag{2-8}$$

where $V \in \Gamma(\phi^{-1}TN)$ and ${}^N R$ is the curvature tensor of (N, h) given by

$${}^N R(X, Y) Z = {}^N\nabla_{[x,y]} Z - {}^N\nabla_X {}^N\nabla_Y Z + {}^N\nabla_Y {}^N\nabla_X Z \tag{2-9}$$

for $X, Y, Z \in \chi(M)$. Now let us consider the following Dirichlet eigenvalue problems of J_ϕ; In case of a compact manifold $\Omega = M$ without boundary,

$$J_\phi V = \lambda V, \quad V \in \Gamma(\phi^{-1}TN). \tag{2-10}$$

In case of a relatively compact domain Ω in M,

$$\text{(2-11)} \qquad \begin{cases} J_\phi V = \lambda V & \text{on } \Omega, \\ V = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the Jacobi operater J_ϕ is a second elliptic differential operater on $\Gamma(\phi^{-1}TN)$, both the eigenvalue problems (2-10) and (2-11) have the discrete spectra consisting of the eigenvalues with finite multiplicities. The index of ϕ (resp. the index of ϕ on Ω), denoted by $\mathrm{Ind}(\phi)$ (resp. $\mathrm{Ind}_\Omega(\phi)$), is defined as the sum of the multiplicities of the eigenvalues of (2-10) (resp. (2-11)). The dimension of zero eigenspace of (2-10)(resp. (2-11)) is called the nullity of ϕ (resp. the nullty of ϕ on Ω), denoted by $\mathrm{Null}(\phi)$ (resp. $\mathrm{Null}_\Omega(\phi)$). A harmonic map ϕ from (M, g) (resp. Ω) is called stable (resp. stable on Ω) if $\mathrm{Ind}(\phi) = 0$ (resp. $\mathrm{Ind}_\Omega(\phi) = 0$). If $\mathrm{Ind}(\phi) = \mathrm{Null}(\phi) = 0$ (resp. $\mathrm{Ind}_\Omega(\phi) = \mathrm{Null}_\Omega(\phi) = 0$), then ϕ is said to be *weakly stable* (resp. weakly stable on Ω). Now we consider that the general properties of the index and nullity. A classical Morse-Schoenberg theorem tells us the index and nullity of a geodesic $\gamma : [0, 2\pi] \to (N, h)$ can be estimated as follows;

THEOREM 1. (Morse-Schoenberg) *Assume that the sectional curvature ${}^N K$ of a Riemannian manifold (N, h) satisfies ${}^N K \leq a$ for some positive constant a. Then the index and nullity of a geodesic $\gamma : I = [0, 2\pi] \to (N, h)$ satisfy*

$$\text{(2-12)} \qquad Ind_I(\gamma) + Null_I(\gamma) \leq (n-1)[L\frac{\sqrt{a}}{\pi}],$$

where L is the length of the geodesic γ and $[x]$ denotes the integer part of $x > 0$.

REMARK. *For any geodesic $\tilde{\phi} : I = [0, 2\pi] \to S^n(\frac{1}{\sqrt{a}})$, we see that*

$$Ind_I(\tilde{\phi}) + Null_I(\tilde{\phi}) \leq (n-1)[L\frac{\sqrt{a}}{\pi}],$$

where L is the length of the geodesic $\tilde{\phi}$ and $S^n(\frac{1}{\sqrt{a}})$ is the canonical sphere of constant curvature a.

REMARK. *If $L < \frac{\pi}{\sqrt{a}}$, then the above inequality (2-12) implies that $Ind_I(\gamma) = Null_I(\gamma) = 0$ which ensures the stability of the geodesic γ. These situation can be naturally extended to the case of general harmonic map. In fact, for a relatively compact domain $\Omega \subset M$ satisfying that Ω is sufficiently small, and a harmonic map $\phi : \Omega \subset (M, g) \rightarrow (N, h)$, it is well known that*

$$Ind_\Omega(\phi) = Null_\Omega(\phi) = 0.$$

In connection with this case we now consider more precisely the estimations of the index and nullity of harmonic maps. In order to do this we use the geometric quantity D defined as follows ;

$$D = {}^N R^\phi_\Omega C(M, g)^{-1} \, Vol\,(\Omega)^{2/m} - 1 \tag{2-13}$$

for a relatively compact domain Ω in M. Where ${}^N R^\phi_\Omega$ is defined by

$$ {}^N R^\phi_\Omega = \max_{x \in \Omega} \max_{0 \neq v \in T\phi(x) N} \frac{h(\sum_{i=1}^m {}^N R(\phi_* e_i, v)\phi_* e_i, v)}{h(v, v)}, \tag{2-14}$$

in case of $\Omega = M$ we denote ${}^N R^\phi_M = {}^N R^\phi$ when defined, and $C(M, g)$is the isoperimetric constant depening only on (M, g) which satisfies the following properties ;

The i-th eigenvalue $\lambda_i(\Omega)$ of the Dirichlet problem

$$\begin{cases} \triangle_M u = \lambda u & \text{on} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2-15}$$

of the Laplace-Beltrami operator $\triangle_M = \delta d$ of (M, g) on $C^\infty(M)$ satisfies the following inequalities

$$\lambda_i(\Omega) \geq C(M, g) \, Vol\,(\Omega)^{-\frac{2}{m}} i^{\frac{2}{m}}, \quad i = 1, 2, \cdots . \tag{2-16}$$

For a sectional curvature ${}^N K$ of (N, h) such that ${}^N K \leq a$ for some positive constant a we have

$${}^N R^\phi_\Omega \leq 2aE^\infty(\Omega, \phi), \tag{2-17}$$

where $E^\infty(\Omega, \phi) = \max_{x \in \Omega} e(\phi)(x)$. Moreover if ϕ is an isometric minimal immersion of (M, g) into (N, h), then ${}^N R^\phi_\Omega \leq ma$, where m is the dimension of the manifold M. Therefore the following theorem and corollary are obtained ;

THEOREM 2. *Let Ω be a relatively compact domain in a complete Riemannian manifold M, if $D < 0$, then for every harmonic map $\phi : \Omega \subset (M,g) \to (N,h)$,*

$$Ind_{\Omega}(\phi) = Null_{\Omega}(\phi) = 0. \tag{2-18}$$

COROLLARY 3. *Let Ω be a relatively compact domain in a complete Riemannian manifold M and $\phi : \Omega \subset (M,g) \to (N,h)$ be a harmonic map. Suppose that the sectional curvature ${}^N K$ of (N,h) satisfies ${}^N K \leq a$ for some positive constant a, and if*

$$C(M,g)\, Vol(\Omega)^{-2/m} > 2aE^{\infty}(\Omega,\phi), \tag{2-19}$$

then also the equality (2-18) holds.

Proof. Since $\lambda_1(\Omega) \geq C(M,g)\, Vol(\Omega)^{-2/m}$, we can easily have the conclusion from (2-17) and the Theorem 3.1 of [10].

Theorem 2 and Corollary 3 tell us that $Vol(\Omega)^{-2/m} \to \infty$ and ${}^N R^{\phi}_{\Omega}$ remains bounded according as Ω decreases in M. Thus we see that every harmonic map $\phi : (M,g) \to (N,h)$ is stable if Ω is sufficiently small. We introduce as a valuable estimation of the index and nullity by the geometric quantity D, the following theorem [10] :

THEOREM 4. *Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g), and $\phi : (M,g) \to (N,h)$ any harmonic map. Assume that $D \geq 0$. Then the index and nullity can be sestimated as follows :*

(1) In case of $m = 1, 2$,

$$Ind_{\Omega}(\phi) + Null_{\Omega}(\phi) \leq n(1 + 1/D)^D(1 + D)$$

(2) In case of $m = 2(p+1),\ p \geq 1$,

$$Ind_{\Omega}(\phi) + Null_{\Omega}(\phi) \leq n(1 + 1/D)^D\{1 + P(D)\},$$

where

$$P(D) = (p+1)! \sum_{k=0}^{p} \frac{i}{k!}\{\frac{1}{\log(1+1/D)}\}^{p+1-k}.$$

(3) In case of $m = 2p+1,\ p \geq 1,$

$$Ind_{\Omega}(\phi) + Null_{\Omega}(\phi) \leq n(1+1/D)^{D}\{1+Q(D)\},$$

where

$$Q(D) = \frac{2}{m}p!\sum_{k=0}^{p}\frac{i}{k!}\{\frac{1}{\log(1+1/D)}\}^{p+1-k}.$$

(4) In case of $m \geq 1,$

$$Ind_{\Omega}(\phi) + Null_{\Omega}(\phi) \leq n\frac{\Gamma(\frac{2}{m}+1)e^{m/2}}{(\frac{m}{2})^{m/2}}(1+D)^{m/2}.$$

REMARK. *$P(D)$, $Q(D)$ satisfy $\lim_{D\to 0} P(D) = \lim_{D\to 0} Q(D)$ and*

$$P(D) \sim (m/2)!D^{m/2},\ Q(D) \sim (m/2)(\frac{m-1}{2})!D^{\frac{m+1}{2}} \text{as} D \to \infty.$$

From (2-17) , the following corollary is given :

COROLLARY 5. *Assume that the sectional curvature ^{N}K of (N,h) is bounded above by a positive constant a. Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g), and $\phi : (M,g) \to (N,h)$ a harmonic map. Then*

$$\text{(2-20)} \quad \begin{aligned} &Ind_{\Omega}(\phi) + Null_{\Omega}(\phi) \\ &\leq n\Gamma(\frac{2}{m}+1)\{\frac{C(M,g)^{-1}ea}{m}\}^{m/2}E^{\infty}(\Omega,\phi)^{m/2}\,Vol\,(\Omega). \end{aligned}$$

Example. In the case $(M,g) = (\mathbf{R}^m, g_0)$, the standard Euclidean space, since $C(\mathbf{R}^m, g_0) = 4\pi^2\omega_m^{-2/m}$ with $\omega_m = \frac{\pi^{m/2}}{\Gamma(\frac{2}{m}+1)}$, the volume of the unit ball, it is clear that for every harmonic map $\phi : (\mathbf{R}^m, g_0) \supset \Omega \to (N,h)$,

$$Ind_{\Omega}(\phi) + Null_{\Omega}(\phi) \leq n(\frac{ea}{m\pi})^{m/2}E^{\infty}(\Omega,\phi)^{m/2}\,Vol\,(\Omega).$$

3. Pluriharmonic maps.

We inspected partially some notions of harmonic maps in the previous section. We now extend those concepts and assertions to the case of pluriharmonic map. Let us consider the case that a pluriharmonic map as an natural extension of harmonic maps, which satisfies a certain condition "isometric totally geodesic immersion".

DEFINITION 6. *Let $\phi : (M, g) \to (N, h)$ be a smooth map between compact connected Riemannian manifolds (M, g) and (N, h). Let us define the p-energy E_p of ϕ by*

$$E_p(\phi) = \frac{1}{p} \int_M \|d\phi\|^p dv_g, \tag{3-1}$$

where p is a real number such that $p \geq 2$, and dv_g denotes the volume element of M. Then ϕ is called a pluriharmonic (simply p-harmonic) map if it is a critical point of the p-energy E_p.

Let ϕ^*TN be the pull-back bundle induced by ϕ over M from the tangent bundle TN of N, and let ∇^* be a formal adjoint concern with L_2-inner product of the connection ∇ of pull-back bundle ϕ^*TN. Then by a direct computation of the first variation of E_p, we get the following Euler-Lagange equation.

LEMMA 7. *The map $\phi : M \to N$ is pluriharmonic if and only if $\nabla^*(\|d\phi\|^{p-2} d\phi) = 0$.*

REMARK. *If $\phi : M \to N$ is an isometric immersion, then $\|d\phi\|$ is a constant and for an arbitrary number $p \geq 2$, the condition that ϕ is p-harmonic is equivalent to the condition that ϕ is (2)-harmonic.*

We now suggest the second variation formula without calculating ;

LEMMA 8. *([7]) Let (M, g) and (N, h) be compact connected Riemannian manifolds and $\phi : M \to N$ be a smooth map. Then for a smooth 2-parameter variation $\phi_{t,s}$ ($|s| < \epsilon$, $|t| < \epsilon$, $\phi_{0,0} = \phi$) of ϕ, we obtain the following second variation formula of the p-energy E_p;*

$$\left[\frac{\partial^2}{\partial t \partial s} E_p(\phi_{t,s})\right]_{t=s=0} = \int_M \langle w, J_p(v)\rangle dv_g, \tag{3-2}$$

where $w = [\frac{\partial \phi_{t,s}}{\partial s}]_{t=s=0}$, $v = [\frac{\partial \phi_{t,s}}{\partial t}]_{t=s=0}$ *and* J_p *is the Jacobi operator defined by*

$$J_p(v) = \nabla^* \left\{(p-2)\|d\phi\|^{p-4}\langle \nabla v,\ d\phi\rangle d\phi + \|d\phi\|^{p-2} \nabla v\right\} - \|d\phi\|^{p-2} T_r{}^N R(d\phi, v)d\phi, \tag{3-3}$$

where $T_r{}^N R$ *is the trace of the curvature tensor* ${}^N R$ *of* N.

DEFINITION 9. *A pluriharmonic map* ϕ *is called p-stable if for an arbitrary smooth variation* $\phi_t(|t| < \epsilon,\ \phi_0 = \phi)$ *of* ϕ, *the second variation (3-2) is non-negative under the condition that* $t = 0$.

From the stability of identity map for harmonic maps, the following lemma can be obtained.

LEMMA 10. *Let* $\phi : M \to N$ *be a non-constant isometric totally geodesic immersion and* $id_M : M \to M$ *be the identity map of* M. *Then* ϕ *is p-stable if and only if* id_M *is stable.*

Direct consequence of combination of definition 9 and above lemmas induces obviously the following theorem.

THEOREM 11. *Let* (M, g) *and* (N, h) *be compact Riemannian manifolds and let a smooth map* $\phi : M \to N$ *be a non-constant isometric totally geodesic immersion. Then for a large number p,* ϕ *is p-stable if and only if* ϕ *is a minimal immersion and stable.*

An immediate corollary to the above theorem is as follows;

COROLLARY 12. *Let* (M, g) *be any compact Riemannian manifold of dimension* m. *Then the identity map* $id_M : M \to M$ *is p-stable for all* $p \geq m$.

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Department of Mathematics Education
Incheon National University of Education
Incheon 403-050, Korea