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WEAK DUALITY IN MULTIOBJECTIVE OPTIMIZATION WITH SET FUNCTIONS

JUN YULL LEE

1. Multiobjective Programming Problem with Set Functions.

Let $(X, \mathcal{A}, \mu)$ be a finite, atomless measure space and $L^1(X, \mathcal{A}, \mu)$ be separable. Then,by considering characteristic function χ_Ω of Ω in $\mathcal{A}$, we can embed $\mathcal{A}$ into $L^\infty(X, \mathcal{A}, \mu)$. In this setting for $\Omega, \Lambda \in \mathcal{A}$, and $\alpha \in I = [0,1]$, there exists a sequence, called *a Morris sequence*, $\{\Gamma_n\} \subset \mathcal{A}$ such that

$$\chi_{\Gamma_n} \xrightarrow{w^*} \alpha\chi_\Omega + (1-\alpha)\chi_\Lambda,$$

where $\xrightarrow{w^*}$ denotes the *weak**– convergence of elements in $L^\infty(X, \mathcal{A}, \mu)$ [4].

A subfamily $\mathcal{S}$ is said to be *convex* if for every $(\alpha, \Omega, \Lambda) \in I \times \mathcal{S} \times \mathcal{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$ in $\mathcal{A}$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathcal{S}$. In ref.[1], if $\mathcal{S} \subseteq \mathcal{A}$ is convex, then the *weak**–closure $cl(\mathcal{S})$ of $\chi_\mathcal{S}$ in $L^\infty(X, \mathcal{A}, \mu)$ is the *weak**–closed convex hull of $\chi_\mathcal{S}$, and $\overline{\mathcal{A}} = \{f \in L^\infty : 0 \leq f \leq 1\}$.

DEFINITION 1.1. *Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$. Let K be a convex cone of R^n. A set function $H : \mathcal{S} \longrightarrow R^n$ is called K-convex, if given $(\alpha, \Omega_1, \Omega_2) \in I \times \mathcal{S} \times \mathcal{S}$ and Morris-sequence $\{\Gamma_n\}$ in $\mathcal{A}$ associated with $(\alpha, \Omega_1, \Omega_2)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n)$ in $\mathcal{S}$ such that*

$$\limsup_{k\to\infty} H(\Gamma_{n_k}) \leqq_K \alpha H(\Omega_1) + (1-\alpha)H(\Omega_2),$$

where lim sup is taken over each component. And $x <_K y$ denotes $y - x \in int(K)$, $x \leq_K y$ denotes $y - x \in K \setminus \{0\}$, and $x \leqq_K y$ denotes $y - x \in K$.

Definition 1.2. *A set function $H = (H_1, \ldots, H_n)\colon \mathcal{S} \to R^n$ is called weak*–continuous on $\mathcal{S}$ if for each $f \in cl(\mathcal{S})$ and for each $j = 1, 2, \ldots, n$, the sequence $\{H_j(\Omega_k)\}$ converges to the same limit for all $\{\Omega_k\}$ with $\chi_{\Omega_k} \xrightarrow{w^*} f$.*

Now multiobjective programming problem with set functions can be described as follows:

$$\text{(P)} \qquad \begin{aligned} &Min_D F(\Omega) \\ &\text{subject to } \Omega \in \mathcal{S} \\ &\text{and } G(\Omega) \leqq_Q \mathbf{0}. \end{aligned}$$

This problem (P) has been defined as the problem finding all feasible efficient $D-$ or properly efficient $D-$solution with respect to the pointed closed convex cones D and Q of R^p and R^m with nonempty interiors, respectively. That is, letting $\mathcal{S}' = \{\Omega \in \mathcal{S}\colon G(\Omega) \leqq_Q \mathbf{0}\}$,we want to find $\Omega \in \mathcal{S}$ such that

$$(F(\mathcal{S}') - G(\Omega)) \cap (-D) = \{\mathbf{0}\}, \emptyset \text{ if } \mathbf{0} \notin D$$

or

$$cl(p(F(\mathcal{S}') + D - F(\Omega^*))) \cap (-D) = \{\mathbf{0}\}, \emptyset \text{ if } \mathbf{0} \notin D,$$

where the set $p(B) = \{\alpha y\colon \alpha > 0, y \in B\}$ is the projecting cone for a set $B \subset R^p$.

For the primal problem (P), we assume that F, G are $D-$convex, $Q-$convex, respectively and *weak**–continuous. Under these assumptions we have the Lagrange multiplier theorem as in usual multiobjective optimization problems. The set of $p \times m$ matrices $\{M \in R^{p\times m}\colon MQ \subset D\}$ is denoted by $\mathcal{L}$.

Theorem 1.1 [3]. *Let Ω^* be a properly efficient $D-$solution to the problem (P). If there is $\Omega_o \in \mathcal{S}$ such that $G(\Omega_o) <_Q \mathbf{0}$, then there exists $M^* \in \mathcal{L}$ such that*

(1) $F(\Omega^*) \in Min_D\{F(\Omega) + M^*G(\Omega) : \Omega \in \mathcal{S}\}$

(2) $M^*F(\Omega^*) = \mathbf{0}$.

In fact, $F(\Omega^*) \in Min_D cl(\{F(\Omega) + M^*G(\Omega)\colon \Omega \in \mathcal{S}\})$.

2. Perturbed Problems and Dual Problems.

The primal problem (P) introduced in previous section is embedded into a family of perturbed problems:

$$Min_D F(\Omega)$$
$$\text{subject to } \Omega \in \mathcal{S}$$
$$\text{and } G(\Omega) \leqq_Q u.$$

The generalized Slater's constraint qualification that there exists $\Omega_o \in \mathcal{S}$ such that $G(\Omega_o) <_Q \mathbf{0}$ is assumed in the sequel. We denote by $\mathcal{S}(u)$ the set $\{\Omega \in \mathcal{S} : G(\Omega) \leqq_Q u\}$,and by $Y(u)$ the set $F(\mathcal{S}(u))$.

DEFINITION 2.1. *Perturbed (or primal) maps are defined on R^m by*

$$W(u) = Min_D F(\mathcal{S}(u))$$

and

$$\overline{W}(u) = Min_D cl(F(\mathcal{S}(u))).$$

The original problem (P) can be therefore regarded as determining $F^{-1}(W(\mathbf{0})) \cap \mathcal{S}$. However, more satisfactory results are obtained if $\overline{W}$ is used instead.

THEOREM 2.1. *The map $\overline{W}$ is a $D-$convex point-to-set map on the convex set $\{u \in R^m : \{\Omega \in \mathcal{S} : G(\Omega) <_Q u\} \neq \emptyset\}$.*

Proof. It is similar to that of [2, Theorem 4.4].

For each $M \in \mathcal{L} = \{M \in R^{p\times m} : MQ \subset D\}$, we define certain maps for (P) on $\mathcal{L}$ by

$$\Phi(M) = Min_D\{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\}$$

$$\overline{\Phi}(M) = Min_D cl(\{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\})$$

The map Φ and $\overline{\Phi}$ are called dual maps for (P).

Remark 2.2.

(1) $MG(\cdot) : \mathcal{S} \longrightarrow R^p$ is D-convex on $\mathcal{S}$.

(2) $L(\cdot, M) = F(\cdot) + MG(\cdot)$ is D–convex and w^*–continuous.

(3) $cl(\{F(\mathcal{S}) + MG(\Omega)\})$ is a D–convex subset of R^p

(4) For each $M \in \mathcal{L}$, since $cl(\{L(\Omega, M) : \Omega \in \mathcal{S}\})$ is compact and D-convex, we have that

$$cl(\{L(\Omega, M) : \Omega \in \mathcal{S}\}) + D = \overline{\Phi}(M) + D.$$

(5) For any u with $\mathcal{S}(u) \neq \emptyset$,$[clY(u)] + D = \overline{W}(u) + D$.

The relationship between the primal map $\overline{W}$ and the dual map $\overline{\Phi}$ now can be established.

THEOREM 2.3. *For any $M \in \mathcal{L}$, the following equalities hold.*

$$\overline{\Phi}(M) = Min_D \bigcup_{u \in \zeta} (\overline{W}(u) + Mu) = Min_D \bigcup_{u \in \zeta^o} (\overline{W}(u) + Mu)$$

where $\zeta = \{u \in R^m \colon \mathcal{S}(u) \neq \emptyset\}$ and $\zeta^o = \{u \in R^m \colon \{\Omega \in \mathcal{S} \colon G(\Omega) <_Q u\} \neq \emptyset\}$.

Proof. Let $y \in \overline{\Phi}(M) = cl\{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\}$. Then there exists a sequence $\{\Omega_n\}$ in $\mathcal{S}$ such that $F(\Omega_n) + MG(\Omega_n) \to y$. Since $cl(F(\mathcal{S}))$ and $cl(G(\mathcal{S}))$ are compact, there exists a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ such that both $F(\Omega_{n_k})$ and $G(\Omega_{n_k})$ converge. Write $\lim_{k\to\infty} F(\Omega_{n_k}) = w'$ and $\lim_{k\to\infty} G(\Omega_{n_k}) = u'$. Then $y = w' + Mu'$. Let $q >_Q \mathbf{0}$. Then $w' \in clY(u' + q) = \overline{W}(u' + q) + D$,by Remark 2.2.(5). Since $MQ \subset D$, it follows that $y + Mq = w' + M(u' + q) \in (\overline{W}(u' + q) + M(u' + q)) + D$. Hence,

$$\text{(i)} \qquad \overline{\Phi}(M) + D \subset \bigcup_{u \in \zeta} (\overline{W}(u) + Mu) + D.$$

Now we suppose that $y \in \overline{W}(u) + Mu$ for some $u \in \zeta$. Then $y - Mu \in \overline{W}(u) = Min_D cl(Y(u)) \subset cl(Y(u))$. Therefore, there is a sequence $\{\Omega_n\}$ in $\mathcal{S}$ such that for any n, $G(\Omega_n) \leqq_Q u$ and $\lim_{n\to\infty} F(\Omega_n) = y - Mu$. Since $cl(G(\mathcal{S}))$ is compact, there exists a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ such that $\{G(\Omega_{n_k})\}$ converges. It follows that

$$y >_D \lim_{k\to\infty} [F(\Omega_{n_k}) + MG(\Omega_{n_k})].$$

Hence, $y \in cl(\{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\}) + D$. Therefore,

$$\text{(ii)} \qquad \bigcup_{u \in \zeta} (\overline{W}(u) + Mu) \subset cl\Psi(M) + D$$

where $\Psi(M) = \{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\}$. Consequently, from (i) and (ii),

$$\overline{\Phi}(M) = Min_D cl(\Psi(M)) = Min_D \bigcup_{u \in \zeta} (\overline{W}(u) + Mu).$$

COROLLARY 2.4. *If Ω^* is a properly efficient D-solution to the problem (P) with generalized Slater's constraint qualification, then there exists an $M^* \in \mathcal{L}$ such that*

$$F(\Omega^*) \in \overline{\Phi}(M^*) \cap \Phi(M^*) \subset Min_D[\bigcup_{u \in \zeta} (\overline{W}(u) + M^*u)].$$

proof. The proof is an immediate consequence of Theorems 1.1 and 2.3.

3. Weak Duality.

Following Sawaragi et al.[5], we define a dual programming problem of (P) as follows:

$$\text{(D)} \qquad Max_D \bigcup_{M \in \mathcal{L}} \Phi(M)$$

,where $\Phi(M) = Min_D\{F(\Omega) + MG(\Omega) \colon \Omega \in \mathcal{S}\}$.

The following weak duality theorem can be proven. Recall that $\mathcal{S}' = \{\Omega \in \mathcal{S} \colon G(\Omega) \leqq_Q \mathbf{0}\}$ denotes the feasible family.

THEOREM 3.1(WEAK DUALITY THEOREM). *Let $M \in \mathcal{L}$. Then for each $\Omega^* \in \mathcal{S}'$ and $y \in \Phi(M)$, it is true that $F(\Omega^*) \not\leq_D y$.*

Proof. Since $G(\Omega^*) \leqq_Q \mathbf{0}$ and $M \in \mathcal{L}$, it follows that $MG(\Omega^*) \leqq_D \mathbf{0}$ and $F(\Omega^*) + MG(\Omega^*) \leqq_D F(\Omega^*)$. Thus, for $y = F(\Omega) + MG(\Omega) \in Min_D\{F(\Omega) + MG(\Omega) : \Omega \in \mathcal{S}\}$, $F(\Omega^*) + MG(\Omega^*) \not\leq_D y$, any $\Omega^* \in \mathcal{S}'$. Therefore, from Lemma 2.3.3[5], $F(\Omega^*) \not\leq_D y$.

THEOREM 3.2. *(1) If $\Omega^* \in \mathcal{S}', M^* \in \mathcal{L}$ and $F(\Omega^*) \in \Phi(M^*)$,then $F(\Omega^*)$ is efficient to (P) and also to (D).*

(2) If Ω^* is properly efficient to (P) and generalized Slater constraint qualification holds for (P), then $F(\Omega^*)$ is effient to dual program (D).

Proof. (1) Suppose that $F(\Omega^*)$ is not efficient to (P). Then $F(\Omega') \leq_D F(\Omega^*)$ for some $\Omega' \in \mathcal{S}'$. Thus, $F(\Omega') + MG(\Omega') \leq_D F(\Omega^*)$, contrary to the assumption that $F(\Omega^*) \in \Phi(M^*) = Min_D\{F(\Omega) + M^*G(\Omega) : \Omega \in \mathcal{S}\}$. Suppose now that $F(\Omega^*)$ is not efficient to (D). Consequently there exists $y \in \Phi(M)$ for some $M \in \mathcal{L}$ such that $F(\Omega^*) \leq_D y$,whence $F(\Omega^*) + MG(\Omega^*) \leq_D F(\Omega^*) \leq_D y$, contrary to $y \in \Phi(M)$.

(2) By Corollary 2.4 $F(\Omega^*) \in \bigcup_{M \in \mathcal{L}} \Phi(M)$, say $F(\Omega^*) \in \Phi(M^*)$ by some $M^* \in \mathcal{L}$. Then by (1), $F(\Omega^*)$ is efficient to the dual problem (D).

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Department of Mathematics Education
Kangweon National University
Chuncheon 200-701, Korea