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AN EXTENDED JIANG SUBGROUP AND ITS REPRESENTATION

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0. Introduction.

F.Rhodes [4] introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as a generalization of the fundamental group of a topological space X and showed that $\sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0) \times G$ if (G, G) admits a family of preferred paths at e. B.J. Jiang [3] introduced the Jiang subgroup $J(f, x_0)$ of the fundamental group of a topological space X.

In this paper, we introduce an extended Jiang subgroup $J(f, x_0, G)$ of the fundamental group of a transformation group as a generalization of the Jiang subgroup $J(f, x_0)$ and give a necessary and sufficent conditon for $J(f, x_0, G)$ to be isomorphic to $J(f, x_0) \times G$.

1. Preliminaries and main results.

Let (X, G, π) be a transformation group, where X is a path connected space with x_0 as base point. Given any element g of G, a path f of order g with base point x_0 is a continuous map $f : I \longrightarrow X$ such that $f(0) = x_0$ and $f(1) = gx_0$. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1 + g_1 f_2$ of order $g_1 g_2$ defined by the equations

$$(f_1 + g_1 f_2)(s) = \begin{cases} f_1(2s), & 0 \le s \le 1/2 \\ g_1 f_2(2s-1), & 1/2 \le s \le 1. \end{cases}$$

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Two paths f and f' of the same order g are said to be homotophic if there is a continuous map $F : I^2 \longrightarrow X$ such that

$$\begin{aligned} F(s,0) &= f(s), && 0 \le s \le 1, \\ F(s,1) &= f'(s), && 0 \le s \le 1, \\ F(0,t) &= x_0, && 0 \le t \le 1, \\ F(1,t) &= gx_0, && 0 \le t \le 1. \end{aligned}$$

The homotopy class of a path f of order g was denoted by $[f \ : \ g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1 x_0 = g_2 x_0$. F. Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group, where $*$ is defined by $[f_1 \ : \ g_1] * [f_2 \ : \ g_2] = [f_1 + g_1 f_2 \ : \ g_1 g_2]$. This group was denoted by $\sigma(X, x_0, G)$, and was called the fundamental group of (X, G) with base point x_0.

Let f be a self-map of X. A homotopy $H : X \times I \longrightarrow X$ is called an f- cyclic homotopy [3] if $H(x,0) = H(x,1) = f(x)$. This concept of a topological space is generalized to that of a transformation group. A continuous map $H : X \times I \longrightarrow X$ is called an f–homotopy of order g if $H(x,0) = f(x)$, $H(x,1) = gf(x)$, where g is an element of G. If H is an f–homotopy of order g, then the path $\alpha : I \longrightarrow X$ given by $\alpha(t) = H(x_0, t)$ will be called the trace of H.

The trace subgroup of f–homotopies of prescribed order is defined by $J(f, x_0, g) = \{[\alpha \ : \ g] \in \sigma(X, f(x_0), G) \mid \text{there exists an } f\text{–homotopy of order } g \text{ with trace } \alpha\}$.

In particular, $J(1_X, x_0, G)$ was defined by $E(X, x_0, G)$ in [5] and $J(f, x_0, \{e\})$ was also defined by $J(f, x_0)$ in [3]. From this fact, we say that $J(f, x_0, G)$ is an extended Jiang subgroup.

It is easy to show that an extended Jiang subgroup $J(f, x_0, G)$ is a subgroup of $\sigma(X, f(x_0), G)$.

Let (X, G) be a transformation group and X^X be the space of all continuous mappings from X to X with compact-open topology. Let G act on X^X continuously by $\pi'(f, g) = gf$. Then (X^X, G, π') is a transformation group .

Let $P : X^X \longrightarrow X$ be the evaluation map given by $P(f) = f(x_0)$. If X is a locally compact,then the evaluation map P is continuous. Since $P(gf) = gf(x_0) = gP(f)$, where $g \in G$ and $f \in X^X, (P, 1_G)$:

$(X^X, G) \longrightarrow (X, G)$ is a category mapping. Thus we know that $P_* : \sigma(X^X, 1_X, G) \longrightarrow \sigma(X, x_0, G)$ defined by $P_*[\alpha : g] = [P \circ \alpha : g]$ is a homomorphism

There is a natural homeomorphism $\phi : (X^X)^I \longrightarrow X^{X \times I}$ given by $\phi(f)(x, s) = f(s)(x)$ for $x \in X$ and $s \in I$.

Note that $f \sim f'$ if and only if $\phi(f) \sim \phi(f')$. Motivated by the following theorem, we can consider $J(f, x_0, G)$ as a generalized evaluation subgroup of the fundamental group of a transformation group (X, G).

THEOREM 1. *Let X be a pathwise connected CW-complex. Then $P_*\sigma(X^X, f, G) = J(f, x_0, G)$.*

Proof. By the above remark, the path $\alpha : I \longrightarrow X^X$ of order g with base point f corresponds to the f–homotopy $\phi(\alpha) : X \times I \longrightarrow X$ of order g.

For every element $[\alpha : g] \in \sigma(X^X, f, G)$, $P_*[\alpha : g] = [P \circ \alpha : g]$ and there exists an f–homotopy $\phi(\alpha)$ of order g with trace $P \circ \alpha$. Thus $P_*[\alpha : g] \in J(f, x_0, G)$.

Conversely, for each element $[\alpha : g]$ of $J(f, x_0, G)$, there exists an f–homotopy $F : X \times I \longrightarrow X$ of order g with trace α. Since $\phi : (X^X)^I \longrightarrow X^{X \times I}$ is a homeomorphism such that $\phi(f)(x, s) = (f(s))(x), \phi^{-1}(F)$ is a path of order g with base point f in X^X, for $\phi^{-1}(F) : I \longrightarrow X^X$ such that $\phi^{-1}(F)(0)(x) = F(x, 0) = f(x)$ and $\phi^{-1}(F)(1)(x) = F(x, 1) = gf(x)$. Thus $[\phi^{-1}(F) : g]$ belongs to $\sigma(X^X, f, G)$. Since $P \circ \phi^{-1}(F)(s) = \phi^{-1}(F)(s)(x_0) = F(x_0, s) = \alpha(s)$, we have $[\alpha : g] \in P_*\sigma(X^X, f, G)$. This completes the proof.

The Jiang's result ([3], Lemma 2.1) can be generalized as follows.

THEOREM 2. *Let f and k be self maps of X.*

(1) *$J(k, f(x_0), G) \subset J(k \circ f, x_0, G)$.*
(2) *If k is a homomorphism of (X, G), i.e., $kg(x) = gk(x)$ for any element g of G, then $k_\pi(J(f, x_0, G)) \subset J(k \circ f, x_0, G)$ where $k_\pi[\alpha : g] = [k\alpha : g]$ for any element $[\alpha : g]$ of $J(f, x_0, G)$.*

Proof. (1). Let $[\alpha : g]$ be an element of $J(k, f(x_0), G)$. Then there exists an k–homotopy $H : X \times I \longrightarrow X$ of order g such that $H(x, 0) = k(x), H(x, 1) = gk(x)$ and $H(f(x_0), t) = \alpha(t)$. Therefore there exists a homotopy $H' = H \circ (f \times 1_I) : X \times$

$I \longrightarrow X$ such that $H'(x,0) = H(f(x),0) = kf(x), H'(x,1) = H(f(x),1) = gkf(x)$ and $H'(x_0,t) = H(f(x_0),t) = \alpha(t)$. Thus $[\alpha : g]$ belongs to $J(k \circ f, x_0, G)$.

(2). Since $k : (X,G) \longrightarrow (X,G)$ is a homomorphism, k induces a homomorphism $k_\sigma : \sigma(X, f(x_0), G) \longrightarrow \sigma(X, kf(x_0), G)$. Let $[\alpha : g]$ be an element of $J(f, x_0, G)$. Then there exists an f–homotopy $H : X \times I \longrightarrow X$ of order g such that $H(x,0) = f(x), H(x,1) = gf(x)$ and $H(x_0,t) = \alpha(t)$. Therefore there exists a homotopy $K = k \circ H : X \times I \longrightarrow X$ such that $K(x,0) = k \circ H(x,0) = kf(x)$, $K(x,1) = k \circ H(x,1) = kgf(x) = gkf(x)$ and $K(x_0,t) = kH(x_0,t) = k\alpha(t)$.

Thus $k_\sigma[\alpha : g]$ belongs to $J(k \circ f, x_0, G)$. Therefore, we show that

$$k_\sigma(J(f, x_0, G)) \subset J(k \circ f, x_0, G).$$

COROLLARY 3 [3]. *Let f and k be selfmaps of X. Then*

(1) $J(k, f(x_0)) \subset J(k \circ f, x_0)$,
(2) $k_\pi(J(f, x_0)) \subset J(k \circ f, x_0)$.

If we take a map $i_* : J(f, x_0) \longrightarrow J(f, x_0, G)$ such that $i_*[\alpha] = [\alpha : e]$, then we can identify $J(f, x_0)$ as a subgroup of $J(f, x_0, G)$. In this case, $J(f, x_0)$ is a normal subgroup of $J(f, x_0, G)$.

In [4], F.Rhodes showed that if λ is a path from x_0 to x_1, then λ induces an isomorphism $\lambda_* : \sigma(X, x_0, G) \longrightarrow \sigma(X, x_1, G)$ such that $\lambda_*[\alpha : g] = [\lambda\rho + \alpha + g\lambda : g]$.

THEOREM 4. *Assume that X is a pathwise connected CW-complex. Let (X,G) be a transformation group. If λ is a path from x_0 to x_1 in X, then the induced homomorphism $(f\lambda)_*$ carries $J(f, x_0, G)$ isomorphically onto $J(f, x_1, G)$.*

Proof. Since $(f\lambda)_* : \sigma(X, f(x_0), G) \longrightarrow \sigma(X, f(x_1), G)$ is an isomorphism, it is sufficient to show that $(f\lambda)_*(J(f, x_0, G))$ is a subset of $J(f, x_1, G)$.

Let $[\alpha : g]$ be any element of $J(f, x_0, G)$. Then there exists an f–homotopy $W : X \times I \longrightarrow X$ of order g with trace

α. Consider a homotopy $H : X \times 0 \bigcup x_1 \times I \longrightarrow X$ given by $H(x,0) = x$ and$H(x_1,t) = \lambda\rho(t)$. Then there exists a homotopy $\tilde{H} : X \times I \longrightarrow X$ such that $\tilde{H}(x,0) = x$ and $\tilde{H}(x_1,t) = H(x_1,t) = \lambda\rho(t)$. Define $K : X \times I \longrightarrow X$ by

$$K(x,t) = \begin{cases} f\tilde{H}(x,3t), & 0 \le t \le 1/3 \\ W(\tilde{H}(x,1), 3t-1), & 1/3 \le t \le 2/3 \\ gf\tilde{H}(x,3(1-t)), & 2/3 \le t \le 1. \end{cases}$$

Then K is an f–homotopy of order g, for

$$\begin{aligned} K(x_1,t) &= \begin{cases} f\tilde{H}(x_1,3t), & 0 \le t \le 1/3 \\ W(\tilde{H}(x_1,1), 3t-1), & 1/3 \le t \le 2/3 \\ gf\tilde{H}(x_1,3(1-t)), & 2/3 \le t \le 1 \end{cases} \\ &= \begin{cases} f\lambda\rho(3t), & 0 \le t \le 1/3 \\ \alpha(3t-1), & 1/3 \le t \le 2/3 \\ gf\lambda(3t-2), & 2/3 \le t \le 1 \end{cases} \\ &= [f\lambda\rho + \alpha + gf\lambda](t). \end{aligned}$$

Thus $(f\lambda)_*([\alpha : g]) = [f\lambda\rho + \alpha + gf\lambda : g]$ belongs to $J(f,x_1,G)$. So, the induced homomorphism $(f\lambda)_*$ is an isomorphism from $J(f,x_0,G)$ to $J(f,x_1,G)$.

THEOREM 5. *If two functions $f,k : X \longrightarrow X$ are homotopic, then $J(f,x_0,G)$ and $J(k,x_0,G)$ are isomorphic.*

Proof. Let $H : X \times I \longrightarrow X$ be a homotopy from f to k and $P(t) = H(x_0,t)$. Then P is a path from $f(x_0)$ to $k(x_0)$. It is sufficient to show that $P_\sigma(J(f,x_0,G)) \subset J(k,x_0,G)$.

Let $[\alpha : g]$ be any element of $J(f,x_0,G)$. Then there exists a homotopy $W : X \times I \longrightarrow X$ such that $W(x,0) = f(x), W(x,1) = gf(x)$ and $W(x_0,t) = \alpha(t)$. If we define a homotopy $K : X \times I \longrightarrow X$ given by

$$K(x,t) = \begin{cases} H(x,1-3t), & 0 \le t \le 1/3 \\ W(x,3t-1), & 1/3 \le t \le 2/3 \\ gH(x,3t-2), & 2/3 \le t \le 1, \end{cases}$$

then $K(x,0) = H(x,1) = k(x)$, $K(x,1) = gH(x,1) = gk(x)$ and

$$K(x_0,t) = \begin{cases} H(x_0, 1-3t), & 0 \le t \le 1/3 \\ W(x_0, 3t-1), & 1/3 \le t \le 2/3 \\ gH(x_0, 3t-2), & 2/3 \le t \le 1. \end{cases}$$

Therefore $[P\rho + \alpha + gP : g]$ belongs to $J(k, x_0, G)$. Therefore $P_\sigma(J(f, x_0, G))$ is contained in $J(k, x_0, G)$.

COROLLARY 6. *If $f, k : X \longrightarrow X$ are homotophic, then $J(f, x_0)$ and $J(k, x_0)$ are isomorphic.*

THEOREM 7. *If $f : X \longrightarrow X$ is a homeomorphism, k is a self map of X and $f(x_0) = k(x_0)$, then $J(f, x_0, G)$ is contained in $J(k, x_0, G)$.*

Proof. Let $[\alpha : g]$ be any element of $J(f, x_0, G)$. Then there exists an f–homotopy $H : X \times I \longrightarrow X$ of order g with trace α. If we define $K : X \times I \longrightarrow X$ be a homotopy such that $K = H \circ (f^{-1}k \times 1_I)$, then

$$K(x,0) = H(f^{-1}k(x), 0) = k(x),$$
$$K(x,1) = H(f^{-1}k(x), 1) = gk(x)$$

and

$$K(x_0, t) = H(f^{-1}k(x_0), t) = H(f^{-1}f(x_0), t)$$
$$= H(x_0, t) = \alpha(t).$$

Therefore $[\alpha : g]$ belongs to $J(k, x_0, G)$.

COROLLARY 8.

(1) *If $f, k : X \longrightarrow X$ are homeomorphisms and $f(x_0) = k(x_0)$, then $J(f, x_0, G)$ is equal to $J(k, x_0, G)$. In particular, $J(f, x_0)$ is also equal to $J(k, x_0)$ for homeomorphisms f and k.*
(2) *If $f : X \longrightarrow X$ is a homeomorphism and $f(x_0) = x_0$, then $J(f, x_0, G)$ is equal to $E(X, x_0, G)$.*

In [4], a transformation group (X, G) is said to admit a family K of preferred paths at x_0 if it is possible to associate with every element g of G a path k_g from gx_0 to x_0 such that the path k_e associated with the identity element e of G is $\hat{x_0}$ which is the constant map such that $\hat{x_0}(t) = x_0$ for each $t \in I$ and for every pair of elements g, h, the path k_{gh} from ghx_0 to x_0 is homotopic to $gk_h + k_g$.

DEFINITION 1. *A family K of preferred paths at $f(x_0)$ is called a family of preferred f-traces at x_0 if for every preferred path k_g in K, $k_g\rho$ is the trace of f-homotopy of order g.*

THEOREM 9. *Let (X, G, π) be a transformation group. If (G, G) admits a family of preferred paths at e, then (X, G) admits a family of preferred f-traces at x_0 for any self map f of X.*

Proof. Let H be a family of preferred paths at e in (G, G). Define $K = \{k_g | k_g(t) = h_g(t)(f(x_0)), h_g \in H\}$. Let $F : X \times I \longrightarrow X$ be the map such that

$$F(x,t) = \pi(f(x), h_g\rho(t)),\ \rho(t) = 1 - t.$$

So,

$$F(x,0) = \pi(f(x), h_g(1)) = h_g(1)f(x) = f(x),$$
$$F(x,1) = \pi(f(x), h_g(0)) = h_g(0)f(x) = gf(x)$$

and

$$F(x_0,t) = \pi(f(x_0), h_g\rho(t)) = h_g\rho(t)f(x_0) = k_g\rho(t).$$

Thus, F is a f-homotopy of order g with trace $k_g\rho$. So, K is a family of preferred f-traces at x_0.

LEMMA 10. *Let (X, G) be a transformation group and let $f : X \longrightarrow X$ be a self map. If k is a trace of a f-homotopy of order g, then for every loop α at x_0, $f\alpha$ is homotopic to*

$k+gf\alpha+k\rho$. In particular, if f is a homeomorphism and α is a loop at $f(x_0)$, α is homotopic to $k+g\alpha+k\rho$.

Proof. Let $H : X\times I \longrightarrow X$ be a f-homotopy of order g with trace k and α be a loop at x_0. Define $F : I\times I \longrightarrow X$ by

$$F(s,t)=\begin{cases} k(4s), & 0\le s\le t/4 \\ H(\alpha((4s-t)/(4-2t)),t), & t/4\le s\le (4-t)/4 \\ k\rho(4s-3), & (4-t)/4\le s\le 1. \end{cases}$$

Then F is well defined and

$$F(s,0)=H(\alpha(s),0)=(f\alpha)(s),$$
$$F(s,1)=(k+gf\alpha+k\rho)(s).$$

In particular, suppose that f is a homeomorphism. Define $F : X\times I \longrightarrow X$ by

$$F(s,t)=\begin{cases} k(4s), & 0\le s\le t/4 \\ H(f^{-1}\alpha((4s-t)/(4-2t)),t), & t/4\le s\le (4-t)/4 \\ k\rho(4s-3), & (4-t)/4\le s\le 1. \end{cases}$$

Therefore $F(s,0)=H(f^{-1}\alpha(s),0)=f(f^{-1}\alpha(s))=\alpha(s)$.

$$\begin{aligned} F(s,1) &= \begin{cases} k(4s), & 0\le s\le 1/4 \\ H(f^{-1}(\alpha((4s-1)/2)),1), & 1/4\le s\le 3/4 \\ k\rho(4s-3), & 3/4\le s\le 1, \end{cases} \\ &= \begin{cases} k(4s), & 0\le s\le 1/4 \\ gff^{-1}(\alpha((4s-1)/2)), & 1/4\le s\le 3/4 \\ k\rho(4s-3), & 3/4\le s\le 1, \end{cases} \\ &= \begin{cases} k(4s), & 0\le s\le 1/4 \\ g\alpha((4s-1)/2), & 1/4\le s\le 3/4 \\ k\rho(4s-3), & 3/4\le s\le 1, \end{cases} \\ &= (k+g\alpha+k\rho)(s). \end{aligned}$$

So, α is homotopic to $k+g\alpha+k\rho$.

THEOREM 11. *A transformation group (X, G) admits a family of preferred f-traces at x_0 if and only if $J(f, x_0, G)$ is a split extension of $J(f, x_0)$ by G.*

Proof. Suppose (X, G) admits a family $K = \{k_g | g \in G\}$ of preferred f-traces at x_0. Consider the sequence:

$$O \longrightarrow J(f, x_0) \xrightarrow{i_G} J(f, x_0, G) \xrightarrow{j_G} G \longrightarrow O,$$

where $i_G([\alpha]) = [\alpha : e]$ and $j_G[\alpha : g] = g$.

Since i_G is a monomorphism, j_G is a epimorphism and $Ker\, j_G = Im\, i_G$, the sequence is a short exact sequence. Define $\psi : G \longrightarrow J(f, x_0, G)$ by $\psi(g) = [k_g\rho : g]$. Then ψ is a homomorphism. Indeed,

$$\begin{aligned}
\psi(g_1 g_2) &= [k_{g_1 g_2}\rho : g_1 g_2] \\
&= [(g_1 k_{g_2} + k_{g_1})\rho : g_1 g_2] \\
&= [k_{g_1}\rho + g_1 k_{g_2}\rho : g_1 g_2] \\
&= [k_{g_1}\rho : g_1] * [k_{g_2}\rho : g_2] \\
&= \psi(g_1) * \psi(g_2).
\end{aligned}$$

By definition of ψ, we have $j_G \circ \psi = 1_G$. Thus $J(f, x_0, G)$ is a split extension of $J(f, x_0)$ by G.

Conversely, suppose $J(f, x_0, G)$ is a split extension of $J(f, x_0)$ by G. Then there is a monomorphism $\psi : G \longrightarrow J(f, x_0, G)$ such that $j_G \circ \psi = 1_G$.

Let $H = \{\alpha_g | \alpha_g\rho$ is a representation path of $\psi(g)\}$. Since $\psi(e) = [\hat{f}(x_0) : e]$ and $\psi(g_1 g_2) = \psi(g_1) * \psi(g_2)$, α_g is a path from $gf(x_0)$ to $f(x_0)$ for each element g of G and $\alpha_e = \hat{f}(x_0)$ and $\alpha_{g_1 g_2}$ is homotopic to $g_1\alpha_{g_2} + \alpha_{g_1}$. So, H is a family of preferred f-traces at x_0. Therefore, a transformation group (X, G) admits a family of preferred f-traces at x_0.

THEOREM 12. *Let $f : X \longrightarrow X$ be a homeomorphism. A transformation group (X, G) admits a family of preferred*

f-traces at x_0 if and only if there exists an isomorphism ϕ : $J(f, x_0, G) \longrightarrow J(f, x_0) \times G$ such that the diagram commutes

$$\begin{array}{ccccccc} & & & J(f,x_0,G) & & & \\ & & \nearrow & & \searrow & & \\ O & \longrightarrow & J(f,x_0) & \downarrow \phi & G & \longrightarrow & O. \\ & & \searrow & & \nearrow & & \\ & & & J(f,x_0)\times G & & & \end{array}$$

Proof. Let $K = \{k_g | g \in G\}$ be a family of preferred f-trace at x_0. Define $\phi : J(f, x_0, G) \longrightarrow J(f, x_0) \times G$ by $\phi([\alpha : g]) = ([\alpha + k_g], g)$. Let $[\alpha : g]$ be an element of $J(f, x_0, G)$. Then there exists a f-homotopy $H : X \times I \longrightarrow X$ such that $H(x, 0) = f(x), H(x, 1) = gf(x)$ and $H(x_0, t) = \alpha(t)$, and $k_g\rho$ is a trace of f-homotopy $J : X \times I \longrightarrow X$ of order g.

Define $F : X \times I \longrightarrow X$ by

$$F(x,t) = \begin{cases} H(x, 2t), & 0 \leq t \leq 1/2 \\ J(x, 2(1-t)), & 1/2 \leq t \leq 1. \end{cases}$$

Then F is a cyclic homotopy with trace $\alpha + k_g$, for

$$F(x,0) = H(x,0) = f(x), \quad F(x,1) = J(x,0) = f(x),$$

$$\begin{aligned} F(x_0,t) &= \begin{cases} H(x_0, t), & 0 \leq t \leq 1/2 \\ J(x_0, 2(1-t)), & 1/2 \leq t \leq 1 \end{cases} \\ &= (\alpha + k_g)(t). \end{aligned}$$

Thus $[\alpha + k_g]$ belongs to $J(f, x_0)$. Let $[\alpha : g] = [\alpha' : g']$. Then α is homotopic to α' and $\alpha + k_g$ is also homotopic to $\alpha' + k_g$. Thus ϕ is well-defined. Suppose $\phi([\alpha : g]) = \phi([\alpha' : g])$. Then $\alpha + k_g$ is homotopic to $\alpha' + k_g$. This implies that $\alpha(= \alpha + k_g + k_g\rho)$ is homotopic to $\alpha'(= \alpha' + k_g + k_g\rho)$. Therefore ϕ is injective.

For any element $([\alpha], g) \in J(f, x_0) \times G$, there exists a cyclic homotopy $H : X \times I \longrightarrow X$ such that $H(x, 0) = f(x) = H(x, 1)$ and $H(x_0, t) = \alpha(t)$.
Since $\{k_g | g \in G\}$ is a family of preferred f-traces at x_0, there exists a f-homotopy $W : X \times I \longrightarrow X$ such that $W(x, 0) = f(x), W(x, 1) = gf(x)$ and $W(x_0, t) = k_g\rho(t)$. Define

$$F(x,t) = \begin{cases} H(x, 2t), & 0 \leq t \leq 1/2 \\ W(x, 2t-1), & 1/2 \leq t \leq 1, \end{cases}$$

then $F(x_0, t) = (\alpha + k_g\rho)(t)$. So, there exists an element $[\alpha + k_g\rho : g]$ in $J(f, x_0, G)$ such that $\phi([\alpha + k_g\rho : g]) = ([\alpha + k_g\rho + k_g], g) = ([\alpha], g)$. Therefore, ϕ is surjective.

Next, we show that ϕ is a homomorphism. Let $[\alpha_1 : g_1]$ and $[\alpha_2 : g_2]$ be elements of $J(f, x_0, G)$. Then

$$\begin{aligned} \phi([\alpha_1 : g_1] * [\alpha_2 : g_2]) &= \phi([\alpha_1 + g_1\alpha_2 : g_1 g_2]) \\ &= ([\alpha_1 + g_1\alpha_2 + k_{g_1 g_2}], g_1 g_2). \end{aligned}$$

while

$$\begin{aligned} \phi([\alpha_1 : g_1]) \circ \phi([\alpha_2 : g_2]) &= ([\alpha_1 + k_{g_1}], g_1) \circ ([\alpha_2 + k_{g_2}], g_2) \\ &= ([\alpha_1 + k_{g_1} + \alpha_2 + k_{g_2}], g_1 g_2). \end{aligned}$$

Since $\alpha_2 + k_{g_2}$ is a loop at $f(x_0)$ and $k_{g_1}\rho$ is a trace of a f-homotopy of order g_1, $\alpha_2 + k_{g_2}$ is homotopic to $k_{g_1}\rho + g_1(\alpha_2 + k_{g_2}) + k_{g_1}$ by Lemma 4-2.

Therefore, we have

$$\begin{aligned} &\alpha_1 + k_{g_1} + \alpha_2 + k_{g_2} \sim \alpha_1 + k_{g_1} + k_{g_1}\rho + g_1(\alpha_2 + k_{g_2}) \\ &+ k_{g_1} \sim \alpha_1 + g_1(\alpha_2 + k_{g_2}) + k_{g_1} \sim \alpha_1 + g_1\alpha_2 + g_1 k_{g_2} \\ &+ k_{g_1} \sim \alpha_1 + g_1\alpha_2 + k_{g_1 g_2}. \end{aligned}$$

This implies that ϕ is a homomorphism.

Conversely, given a commutative diagram with exact rows and ϕ which is an isomorphism :

$$
\begin{array}{ccccccc}
 & & & J(f,x_0,G) & & & \\
 & & {}^{i_G}\nearrow & & \searrow^{j_G} & & \\
O & \longrightarrow & J(f,x_0) & \downarrow \phi & G & \longrightarrow & O. \\
 & & {}_{i_1}\searrow & & \underset{\pi_2}{\swarrow}\overset{i_2}{\nearrow} & & \\
 & & & J(f,x_0)\times G & & &
\end{array}
$$

define $\psi : G \longrightarrow J(f,x_0,G)$ to be $\phi^{-1} \circ i_2$. Use the commutativity of the diagram to show $j_G \circ \psi = 1_G$. Then there is a monomorphism $\psi : G \longrightarrow J(X,x_0,G)$ such that $j_G \circ \psi = 1_G$. So, $J(f,x_0,G)$ is a split extension of $J(f,x_0)$ by G. By Theorem 11, (X,G) admits a family of preferred f-traces at x_0.

COROLLARY 13. *Let $f : X \longrightarrow X$ be a homeomorphism. A transformation group (X,G) admits of preferred f-traces at x_0 and G abelion if and only if $O \longrightarrow J(f,x_0) \longrightarrow J(f,x_0,G) \longrightarrow G \longrightarrow O$ is a split exact sequence of Z-module.*

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