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A STUDY ON THE PRIME MODULES

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1. Introduction.

For any associative ring $R\ (\neq 0)$, R^1 is defined as $R^1 = R$ if $1 \in R$, otherwise $R^1 = \mathbf{Z} \times R$ is the ring with an identity $(1, 0)$, where $\mathbf{Z}$ is the ring of integers and two operations in R^1 are defined as follows : for all $m, n \in \mathbf{Z}$ and $r, s \in R$,

$$(m,r) + (n,s) = (m+n, r+s)$$
$$(m,r) \cdot (n,s) = (mn, ms + nr + rs).$$

NOTATION. *In this paper $A \subset B$ means $A \subseteq B$ but $A \neq B$.*

If we identify R with $\{0\} \times R$, then R is as (two-sided) ideal of R^1. All R-module here will be right R-module unless explicitly stated otherwise. A submodule $K < M$ of an R-module M is prime if for any $n \in M - K$ and $t \in R$ such that $mR^1 t \subseteq K$, necessarily $t \in \mathrm{ann}_R(M/K)$ where $\mathrm{ann}_R(M/K) = \{r \in R \mid Mr \subseteq K\}$ is the annihilator ideal of M/K in R. The submodule $K < M$ is semiprime if for any $m \in M$ and $s \in R$ such that $msR^1 s \subseteq K$, $ms \in K$.

The purpose in this paper is to investigate characterization of prime module with some additional properties, and our main theorems are follows :

THEOREM 3.1. *If M is a uniform module,then M is prime module if and only if every cyclic submodule of M is a prime module.*

THEOREM 3.4. *Let K be an irreducible submodule of an R-module M, then the following are all equivalent :*

(1) *$K < M$ is prime*
(2) *$K < M$ is semiprime*
(3) *$\mathrm{ter}(M/K) \subseteq (M/K)^2 = \mathrm{ann}_R(M/K)$.*

2. Definitions and preliminaries.

In this section, we discuss some definitions and preliminary results that need to prove our main theorems in section 3.

NOTATION. *Right R-submoudles of an R-module are denoted by " $\leq$ ", proper ones by " $<$ ", while "Δ" indicates two - sided ideals of R. For any right R - submoudle $V \leq M$, $\text{ann}_R V = \{a \in R \mid V_r = 0\}\Delta R$. For $m \in M$ and $K \leq M, m^{-1}K = \{r \in R \mid mr \in K\}$ is the right ideal of R. If $K = 0$, then we denote $m^{-1}o = m^{-1}(0)$ as $\text{ann}_R(m)$.*

DEFINITION 2.1. *For any right R-moudle M, a submodule $K \leq M$ is a prime submodule if $\forall \beta \in M - K, t \in R, \beta R^1 t < K$ then t is an anmihilator (M/K) in R. A R-module M is prime if$(0) < M$ is a prime submodule.*

PROPOSITION 2.2. *Let K be a submodule of a right R-module M and let $\text{ann}_R(M/K)$ be the annihilator ideal of M/K in R. Then the following are all equivalent :*

(1) *$K < M$ is prime submodule.*
(2) *$\forall \beta \in M - K, \forall t \in R, \beta Rt \subseteq K \rightarrow t \in \text{ann}_R(M/K)$.*
(3) *Every nonzero submodule of M/K has the same annihilator, i, e, $\text{ann}_R(M/K)$.*
(4) *$\forall V \leq M, \quad \forall A \leq R, \quad VA \subseteq K \rightarrow$ either $V \subseteq K$ or $A \subseteq \text{ann}_R(M/K)$.*
(5) *$\forall K \subset W \leq M, \forall \text{ann}_R(M/K) \subset B \leq R \rightarrow WB \not\subseteq K$.*

Proof. (1) $\Longrightarrow$ (2) : Let $\beta \in M - K$, $t \in R$ with $\beta Rt \subseteq K$. If $\beta R^1 t \subseteq K$, then by(1), $t \in \text{ann}_R(M/K)$. So assume that $\beta R^1 t \not\subseteq K$ and consequently $t \in \text{ann}_R(M/K)$. Take a $\rho \in R^1$ with $\beta\rho t \notin K$. By (1), $(\beta\rho t)R^1 t \not\subseteq K$. Hence there exists an element $\partial \in R^1$ with $\beta\rho t\partial t \notin K$, which contradicts $\beta\rho t\partial t \in \beta Rt \subseteq K$ since $R\Delta R^1$. Thus $\beta R^1 t \subseteq K$, and by(1), $t \in \text{ann}_R(M/K)$.

(2) $\Longrightarrow$ (1) : Let $\beta \in M - K, \quad t \in R$ with $\beta R^1 t \subseteq K$. Then since $\beta Rt \in \beta Rt, \quad \beta Rt \subseteq K \rightarrow t \in \text{ann}R(M/K)$ by (2). Therefore K is a prime submodule.

(1) $\Longrightarrow$ (3) : It sufices to show that $\forall 0 \neq \beta + K \in (M/K)$ with $\beta \in M - K, \text{ann}_R((\beta + K)R) = \text{ann}_R(M/K)$. But if $t \in \text{ann}_R((\beta R^1 +$

$K)/K)$, then $\beta R^1 t + K \subseteq K$ and so $\beta R^1 t \subseteq K$. Hence by(1), $t \in \mathrm{ann}_R(m/k)$. ie, $\mathrm{ann}_R((\beta R^1 + K)/K) \subseteq \mathrm{ann}_R(M/K)$. Since we have $\mathrm{ann}_R(M/K) \subseteq \mathrm{ann}_R((\beta R^1 + K)/K), \mathrm{ann}R((\beta + K)R^1) = \mathrm{ann}_R(M/K)$.

(3) $\Longrightarrow$ (1) : Let $\beta R^1 t \subseteq K$ for some $\beta \in M - K$ and $t \in R$. Then $(\beta R^1 + K)t \subseteq K \rightarrow t \in \mathrm{ann}_R(\beta R^1 + K)/K = \mathrm{ann}_R(M/K)$.

(1) $\Longrightarrow$ (4) : Let $V \leq M, \quad A \leq R$ and $VA \subseteq K$. Then if $V \nsubseteq K$, Then by(1) $VR^1 A \subseteq K \rightarrow A \subseteq \mathrm{ann}R(M/K)$.

(4) $\Longrightarrow$ (1) : Let $\beta R^1 t \subseteq K$ with $B \in M - K$ and $t \in R$, set $V = \beta R^1$ and $A = tR^1$. Since $VA \subseteq K$, but $V \nsubseteq K$, it follows from (4) that $t \in A \subseteq \mathrm{ann}_R(M/K)$.

(4) $\Longrightarrow$ (5) : Let $K \leq W \leq M$ with $K \neq W$ and $\mathrm{ann}_R(M/K) \subset B \leq R$. If $WB \subseteq K$, then either $W \subseteq K$ or $B \subseteq \mathrm{ann}_R(M/K)$, which contradicts.

(5) $\Longrightarrow$ (4) : Let $V \leq M$ and $A \leq R$ with $VA \subseteq K$. Define $W = V + K$ and $B = A + \mathrm{ann}_R(M/K)$. If $V \subseteq K$ then $W \subseteq K$; if $W \subseteq K$, then $V \subseteq K$ thus $V \nsubseteq K = W \nsubseteq K$ and $A \nsubseteq \mathrm{ann}_R(M/K) \leftrightarrow B \nsubseteq \mathrm{ann}_R(M/K)$. Hence $WB \subseteq K$ and so by(5), either $W \leq K$ or $B \subseteq \mathrm{ann}_R(M/K)$. Thus $V \subseteq K$ or $A \subseteq \mathrm{ann}_R(M/K)$. If $K = M$ or $MR \subseteq K$ then $K \leq M$ is clearly prime submodule.

Corollary 2.3. *An R-module M is prime module if and only if for every nonzero submodule N, $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$.*

Proposition 2.4. *If K is a prime submodule of a right R-module, then $\mathrm{ann}_R(M/K)$ is a prime ideal of R.*

Proof. Let $M_s R_t \subseteq K$ with $ms \notin K$ for $m \in M, s, t \in R$. Set $\beta = ms$. Then since $\beta R_t \subseteq K$, by proposition 2.2.(2), $t \in \mathrm{ann}_R(M/K)$. Hence $\mathrm{ann}_r(m/k)$ is a prime ideal of R.

Definition 2.5. *[2] For a submodule $K \leq M$ of an R-module M,let $\beta \in M$ and $s \in R$ be arbitrary. Then $K \leq M$ is called (new) semiprime if $\beta s R^1 s \subseteq K \rightarrow \beta s \in K$.*

PROPOSITION 2.6. *For a submodule $K \leq M$ of an R - module M, the following are equivaient:*

(1) *K is a semiprime submdule.*
(2) $\forall \beta \in M, \quad \forall s \in R, \quad \beta sRs \subseteq K \rightarrow \beta s \in K$.

Proof. (1) $\Longrightarrow$ (2) : Let $\beta sRs \subseteq K, \beta sR^1 s \not\subseteq K$, and $\beta sts \notin K$ for $t \in R^1$. Then $\beta sts R^1 sts \subseteq \beta sRs \subseteq K$. By(1), $\beta sts \in K$, which is a contradiction. Thus $\beta sR^1 s \subseteq K$, and consequently $\beta s \in K$.

(2) $\Longrightarrow$ (1) : Since $\beta sRs \subseteq \beta sR^1 s$, (2) $\rightarrow$ (1) is clear.

DEFINITION 2.7. *An R-module $M(\neq 0)$ is called auniform module if any two nonzero submodules of M have nonzero intersection.*

PROPOSITION 2.8. *An R - module $M \neq 0$ is a uniform if and only if every nonzero submodule N is essential in M, ie, for any nonzero submodule $K \leq M, N \cap K \neq (0)$.*

Proof. The proof is clear.

NOTATION. *$N \ll M$ meass that N is an essential submodule of an R - module M.*

PROPOSITION 2.9. *Let M be an R - module,and let $K_1 \subseteq N_1$ and $K_2 \subseteq N_2$ be four submodules of M.*

(1) *If K_1 and K_2 are essential in M, then so is $K_1 \cap K_2$.*
(2) *if K_1 is essential in M, then so is N_1.*
(3) *If K_1 is essential in N_1 and if N_1 is essential in M, then K_1 is essential in M.*
(4) *If K_1 is essential in N_1, N_2 is essential in N_2 and $N_1 \cap N_2 = 0$, then $K_1 \oplus K_2$ (direct sum) is essential in $N_1 \oplus N_2$.*

Proof. Let X be a nonzero submoudle of M. Then (1),(2), and (3) follow immediately from the three observations $(K_1 \cap K_2) \cap X = K_1 \cap (K_2 \cap X), N_1 \cap X \supseteq K_1 \cap X$ and $K_1 \cap X = K_1 \cap (N_1 \cap X)$. For part (4)we may clearly suppose that $X \subseteq N_1 \oplus N_2$ but that $X \cap N_1 = X \cap N_2 = 0$. Let $x = x_1 + x_2$ with $x_i \in N_i$ be a nonzero element of X. Then $xR \cap K_1 \neq 0$ so that we can multiply x by a suitable element of R to obtain $y = e_1 + y_2 \in X, y \neq 0$ with $e_1 \in K_1, y_2 \in N_2$. Furthermore $y_2 R \cap K_2 \neq 0$ so that multiplying Y by a suitable element of R shows that $X \cap (K_1 \oplus K_2) \neq 0$. For examples, every simple R-module is uniform,and also and nonzero ideal in a commutative integral domain is a uniform module.

3. Main theorems.

In this section we will prove our main theorems .

THEOREM 3.1. *Let R be a ring with identity. If M is a uniform R - module, then M is prime module if and only if every cyclic submodule N of M is a prime module.*

Proof. $(\Rightarrow)$: Since every nonzero submodule L of N is also a submodule of M, we have $\text{ann}_R(L) = \text{ann}_R(M) = \text{ann}_R(N)$ and hence N is a prime module.

$(\Leftarrow)$: By corollary 2.3, it suffices to show that for any $0 \neq V < M, \text{ann}_R(V) = \text{ann}_R(M)$ If not, then $\text{ann}_R(M) \subsetneq \text{ann}_R(V)$ and hence for some $a \in R$ and $x \in M, Va = 0$ but $xa \neq 0$. Take any $0 \neq y \in V$ and any $0 \neq z \in xR \cap yR$. Since xR is a prime module and $zR \subseteq xR$, by cor, 2.3, $\text{ann}R(zR) = \text{ann}R(xR)$. But then $zR \subseteq yR \subseteq V$ implies that $(zR)a = 0$ and so $(xR)a = 0$. Thus $(xR)a \neq 0$ is a contradiction. Hence $\text{ann}_R(V) = \text{ann}_R(M)$ and so M is a prime module. For any R - module W, the tertiary radical of $W[2,3]$ is the ideal ter $W = \{\text{ann}_R V | V \text{is essential in } W\} \Delta R$. A submodule $K < M$ is (meet) irreducible if for any submodule $A, B < M$ such that $K = A \cap B$, either $K = A$ or $K = B$.

LEMMA 3.2. *Let M be an (right) R - module. Then*

(1) *ter M is a two - sided ideal of R.*
(2) *If N is a submodule of M, then ter $N \supseteq$ ter M.*
(3) *If L is an essntial extension of M, then ter M = ter L.*

In particular, if $E(M)$ is an injective hull of M,then ter M = ter $E(M)$.

Proof. (1) If $t, s \in \text{ter}M$, then $Nt = 0$ and $N's = 0$ for some essential submodule N and N' of M, respecively. Since $N \cap N'$ is also an essential submodule,we have $(N \cap N')(t-s) = (N \cap N')t - (N \cap N')s = 0 - 0 = 0$, and so $t - s \in$ ter M. For any $r \in R, N(rt) \subseteq Nt = 0$ and $N(te) = (Nt)r = 0r = 0$, and hence tr and $rt \in$ ter M. Thus ter$M$ is a two - sided ideal of R.

(2) Let $r \in \text{ter } M$. Then $Ar = 0$ for some essential submodule A of M. Since $A \cap N$ is an essential submodle of N and $(A \cap N)r \subseteq Nr = 0$, we have $r \in \text{ter } N$. Thus $\text{ter } N \supseteq \text{ter } M$.

(3) From(2), it remains to show that $\text{ter } M \subseteq \text{ter } L$. If $t \in \text{ter}M$, then $Bt = 0$ for some essential submodule B of M. Since M is essential submodule of L, B is also an essential submodule of L by proposition 2.9. Thus $t \in \text{ter}L$ for all $t \in \text{ter}M$. Hence $\text{ter}M = \text{ter}L$. If $E(M)$ is an injective hull of M, then $E(M)$ is an essential extension of M and so $\text{ter}E(M) = \text{ter}M$. Note that if $K \leq M$ is irreducible, then $\text{ter}(M/K) = \cup\{R^1 : \beta^1 K | \beta \in M - K\}[5, p369]$. For any submodule $K \leq M$,it will be convenient to define $M : K \Delta R$ as the ideal $M : K = \{r \in R | Mr \subseteq K\}$, although $M : K = \text{ann}_R(M/K)$ and $M : (0) = \text{ann}_R(M)$.

Lemma 3.3. *A submodule K of an R - module M is a prime if and only if $M : K = R^1 : \beta^{-1}K$ for any $\beta \in M - K$. Moreover, $\text{ann}_R(M/K)$ is the tertiaryr radical of M/K.*

Proof. If $t \in M : K$, then $Mt \subseteq K$ and so for any $\beta \in M, \beta R^1 t \subseteq Mt \subseteq K$. Thus $t \in R^1 : \beta^{-1}K$. Hence $M : K \subseteq R^1 : \beta^{-1}K$. $M : K \supseteq R^1 : \beta^{-1}K$ if and only if for all $r \in R$, $\beta R^1 r \subseteq K \rightarrow Mr \subseteq K$. This is so for all $\beta \neq K$ if and only if K is prime. Consequently, the first assertion is prooved. Finally, $\text{ter}(M/K) = \cup\{B : K | B/K \text{is essential submodule of } M/K\}$ and $B : K \subseteq R^1 : \beta^{-1}K$ for all $\beta \in B$ since for all $t \in B : K$, $\beta R^1 t \subseteq Bt \subseteq K \rightarrow t \in R^1 : \beta^{-1}K$. Thus $BR^1(B : K) \subseteq K$ and $B \not\subseteq K$. If K is prime, then $B : K \subseteq M : K$. Therefore, $\text{ter}(M/K) \subseteq \text{ann}_R(M/K)$, i.e., $\text{ter}(M/K) = \text{ann}_R(M/K)$ since M/K is essentisl submodule of M/K.

Theorem 3.4. *Let R be an(meet) irreducible submodule of an R - module M. Then the following are all equiviaient :*

(1) *$K < M$ is prime submodule.*
(2) *$K < M$ is semiprime submodule.*
(3) *$\text{ter}(M/K) \subseteq \text{ann}_R(M/K)$.*

proof. (1) $\Rightarrow$ (3): It follows from Lemma 3.3.

(3) $\Rightarrow$ (1): If (3) holds, then $\text{ter}(M/K) \subseteq \text{ann}_R(M/K) = M : K$

implies thet $M : K$ contains $R^1 : \beta^{-1}K$ for all $\beta \in M - K$, and so K is prime since $\beta R^1 t \leq K \rightarrow t \in \text{ann}_R(M/K)$.

(1) $\Rightarrow$ (2): Let $\beta_s R^1 s \leq K$. Then $s \in \text{ann}_R(M/K)$, i.e., $Ms \leq K$ since K is prime. Hence $\beta_s \in K$.

(2) $\Rightarrow$ (1): Assume that K is not prime. Then $\beta R^1 t \leq K$ for some $\beta \in K$ and $t \in \text{ann}_R(M/K)$. Since K is ireducible, $(\beta R^1 + K) \cap (MtR^1 + K)$ is not contained in K. Hence we can find $r \in R^1$ and $\alpha \in M$, such that $\alpha tr \in \beta R^1 + K$ but $\alpha tr \notin K$. Now we have $\alpha tr R^1 tr \subseteq \beta R^1 tr + K \leq K$ and hence K is not semiprime since $\alpha tr \in K$.

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