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ON THE RANK OF THE FLOW

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In this paper a *flow* will be a pair (X,T) where X is a compact Hausdorff space, and T is a homeomorphism of X onto X. We will usually but not always assume that X is a metric space. We sometimes suppress the homeomorphism T notationally, and just denote a flow by X. General references for the preliminary dynamical notations discussed in this section are [6] and [4]. (The latter should be used with care, since transformations are written on the right there.)

The *trivial* flow is the identity homeomorphism on the one-point space; in this case we write $X = 1$.

If $x \in X$, the *orbit* of x is the set $\mathcal{O}(x) = \{T^n(x) \mid n \in Z\}$. A subset Λ of X is a *minimal* set if Λ is nonempty, closed, T-invariant ($T(\Lambda) \subset \Lambda$), and contains no proper subsets with these properties. It is easy to see that a nonempty subset Λ of X is minimal if and only if it is the orbit closure of each of its points: $\Lambda = \overline{\mathcal{O}(x)}$, for all $x \in \Lambda$. If $\overline{\mathcal{O}(x)}$ is a minimal set, x is said to be an *almost periodic point* . If (X,T) is itself minimal, (so $X = \overline{\mathcal{O}(x)}$ for every $x \in X$), then we say that (X,T) is a *minimal flow.*

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If (X,T) and (Y,S) are flows, a *homeomorphism* from X to Y is a continuous equivariant map π, that is, $\pi(T(x)) = S(\pi(x))$, for all $x \in X$. If (Y,S) is minimal, π is necessarily onto.

A flow (X,T) is *equicontinuous* if the powers of $T\{T^n | n \in Z\}$ form an equicontinuous family of maps. Every flow (X,T) has a unique maximal equicontinuous factor - that is, there is an equicontinuous factor of (X,T) such that every equicontinuous factor of (X,T) is a factor of (Y,S) [4].

If (X,T) is a flow, the points x and y are *proximal* if there is a sequence $\{n_i\}$ such that $\lim T^{n_i}(x) = \lim T^{n_i}(y)$. (Here and elsewhere in the paper, when we use sequences in definitions and arguments we are assuming the phase space is metric. In case the space is not metric " sequences " should be replaced by " nets"). Equivalently, x and y are proximal if the diagonal $\delta_X \subset \overline{\mathcal{O}(x,y)}$. We write $P(X,T)$ or just P for the proximal relation: $P = \{(x,y) | x \text{ and } y \text{ are proximal}\}$. It is clear that P is an invariant subset of the flow $(X \times X, Y \times Y)$ and that $P(X,T) = P(X,T^n)$ for $n \neq 0$.

A homeomorphism $\pi : X \to Y$ is said to be proximal if, whenever $\pi(x) = \pi(x')$, then $(x,x') \in P$. In this case, we say that X is a proximal extension of Y.

Given a flow (X,T) and an index set Γ, we consider the flow (X^Γ, T_Γ) on the product space X^Γ in which T acts on each coordinate and $T_\Gamma : X^\Gamma \to X^\Gamma$ is defined by $[T_\Gamma(z)]_\delta = T(z_\delta)$. We will denoted by $z_\Gamma = \{z_\delta \mid \delta \in \Gamma\}$. It is easy to prove the following remark.

REMARK 1.1. *Let* Γ *and* Γ' *be any two index sets and let* $z \in X^{\Gamma}$ *and* $w \in X^{\Gamma'}$ *be such that* $z_{\Gamma} = w_{\Gamma'}$. *Then* z *is an almost periodic point of* (X^{Γ}, T_{Γ}) *if and only if* w *is an almost periodic points of* $(X^{\Gamma'}, T_{\Gamma'})$.

DEFINITION 1.2. *A subset* Λ *of* X *is said to be an* ***almost periodic set*** *if there exists an index set* Γ *and an almost periodic point* z *of* (X^{Γ}, T_{Γ}) *such that* $z_{\Gamma} = \Lambda$.

DEFINITION 1.3. *A subset* Λ *of* X *is an* ***almost periodic set mod T*** *if it is an almost periodic set and distinct points of* Λ *lie on distinct orbits that is* $x, y \in \Lambda$ *and* $x = T^j(y)$ *implies* $j = 0$. *Clearly maximal almost periodic sets mod* T *exist.*

REMARK 1.4. *Let* $z \in X^{\Gamma}$ *and let* $w \in \overline{O(z)}$. *If* $z_{\alpha} = T^j(z_{\delta})$ *for some* $\alpha, \delta \in T$, *then* $w_{\alpha} = T^j(w_{\delta})$.

LEMMA 1.5 ([1]). *All maximal almost periodic sets mod* T *of the flow* (X, T) *have the same cardinality.*

By the Lemma 1.5, we can define the ***almost periodic rank*** (or just rank) of a flow to be the cardinality of a maximal almost periodic set mod T.

In a similar manner we can define the ***almost periodic rank of a homomorphism.*** Let (X, T) and (Y, S) be flows with (Y, S) minimal and $|Y|$ infinite and let $\pi : X \to Y$ be a homomorphism. The rank of

π is defined to be the cardinality of a maximal almost periodic subset of a fiber (that is, rank $\pi = |\Lambda|$, where Λ is an almost periodic subset of $\pi^{-1}(y)$ for $y \in Y$, and Λ is maximal with respect to this property. Since Y is infinite, any such almost periodic set is necessarily almost periodic mod T).

The following theorem certify that the rank of π is well defined.

THEOREM 1.6. *All maximal almost periodic subsets of any fiber of π have the same cardinality.*

proof. Let Λ and Λ' be maximal almost periodic sets of the fibers $\pi^{-1}(y)$ and $\pi^{-1}(y')$. If $y = y'$, choose index sets Γ and Γ' and points $z \in X^{\Gamma}$ and $z' \in X^{\Gamma'}$ such that $\Gamma \cap \Gamma' = \phi, z_{\Gamma} = \Lambda$ and $z'_{\delta'} = \Gamma'$.

Consider $(z, z') \in X^{\Gamma} \times X\Gamma'$ and let $T_{\Gamma} \times T_{\Gamma'}$ act on this space. There exists a $T_{\Gamma} \times T_{\Gamma'}$ minimal set Y in the orbit closure of (z, z') which projects onto $\overline{O(z)}$. Hence there exists a $T_{\Gamma} \times T_{\Gamma'}$ almost periodic of the form (z, w). Set $\Lambda_0 = w_{\Gamma'}$. Since $w \in \overline{O(z)}, \Lambda_0$ is a maximal almost periodic set mod T and $|\Lambda'| = |\Lambda_0|$ and Λ_0 is a maximal almost periodic set, $|\Lambda'| = |\Lambda_0|$ and $\Lambda_0 \subset \pi^{-1}(y)$. Because $\Lambda \cap \Lambda_0$ is an almost periodic set, $|\Lambda| = |\Lambda_0|$ by maximality. If $y \neq y'$, choose $z'' \in \overline{O(z')}$ such that $\Lambda'' = z''_{\Gamma'} \subset \pi^{-1}(y)$ and Λ'' is a maximal almost periodic subset of $\pi^{-1}(y)$. Then $|\Lambda'| = |\Lambda''|$ and by the above $|\Lambda''| = |\Lambda|$.

LEMMA 1.7. *Let (X, T) and (Y, S) be minimal flows and let $\phi : X \to Y$ a homomorphism. Then the following are equivalent :*

(a) *There exist $x, y \in X$, with $x \neq y$, and $\phi(x) = \phi(y)$ such that (x, y) is an almost periodic point of $(X \times X, T \times T)$*

(b) There exist $x, y \in X$, *with* $\phi(x) = \phi(y)$ *such that* $(x, y) \notin P(X, T)$.

We have the following theorem by definition.

THEOREM 1.8. *Let* (X, T) *and* (Y, S) *be infinite minimal flows and let* $\pi : X \to Y$ *be a homomorphism. Then*

(i) rank $(X, T) =$ *(rank* π*)rank* (Y, S)

(ii) rank $(Y, S) \leq$ *rank* (X, T)

We will state main theorem of this paper.

THEOREM 1.9. *Let* (X, T) *be a totally minimal flow and let* (Y, T) *be a nontrivial minimal flow. Let* $\phi : (X, T) \to (Y, T)$ *be a homomorphism. If* (X, T) *is a graphic then rank*$\phi = 1$.

proof. At first, we will show that ϕ is proximal homomorphism. Suppose ϕ is not proximal. Then there exist $x, x' \in X$ with $x \neq x'$ and $\phi(x) = \phi(x')$ such that $(x, x') \in P(X, T)$. By Lemma 1.7, there exist $x, x' \in X$ with $x \neq x'$ and $\phi(x) = \phi(x')$ such that (x, x') is an almost periodic point of $(X \times X, T \times T)$. Since (X, T) is graphic, $x' = T^m(x)$ for some $m \neq 0$. So $\phi(x') = \phi(T^m(x)) = \phi(x)$. Since (X, T) is totally minimal, $X = \overline{\{T^n(x) | n \in Z\}}$. Let $z \in X$. Then there exists a net $\{n_i\}$ in Z such that $T^{n_i}(x) \to Z$ and so $\phi(T^{n_i}(x)) \to \phi(z)$. Hence any $z \in X$, $\phi(z) = \phi(x)$. This implies Y is trivial. This is a contradiction because Y is nontrivial. Therefore ϕ is proximal. Since ϕ is proximal, let $x, x' \in \phi^{-1}(y)$ with $x \neq x'$, then (x, x') is not an almost periodic point of $(X \times X, T \times T)$ because $\Delta \subset \overline{O(x, x')}$, so rank ϕ must be 1.

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