Kangweon–Kyungki Math. Jour. **2** (1994), pp. 7–12

ON THE PETTIS DECOMPOSABILITY AND DUNFORD DECOMPOSABILITY

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1. Introduction

Huff[7] studied Pettis integrability using the associated operator from a Banach space into $L_1(\mu)$. Bator[1] introduced the notion of a Pettis decomposable function from a finite measure space into a dual Banach space which is an extension of a Pettis integrable function.

In this paper, in order to generalize the Huff's results we study Pettis decomposability using the associated operator. Furthermore, we introduce the notion of a Dunford decomposable function which is an extension of a Pettis decomposable function, and obtain some properties of the Dunford decomposable function.

2. Preliminaries

Throughout this paper, let (Ω, Σ, μ) be a finite measure space and let X and Y be real Banach spaces with duals X^* and Y^*, respectively. We denote the closed unit ball of X by B_X. The adjoint of a continuous linear operator T from X to Y will be denoted by T^*.

If $f : \Omega \to X^*$ is a Dunford integrable function, then we define $T_f : X^{**} \to L_1(\mu)$ by $T_f(x^{**}) = x^{**} \circ f$ for each $x^{**} \in X^{**}$ and let $\tilde{T_f} = T_f|_X$, where T_f is called the associated operator with f. Note that if $f : \Omega \to X^*$ is a Dunford equi-integrable function then $\tilde{T_f}^{**}(X^{**}) \subseteq L_1(\mu)$.

We use the following theorems in order to obtain our results.

Received December 21, 1993.
Supported by Kangwon National University Research Grant, 1993.

THEOREM 2.1 [1]. *A weak* scalarly bounded function $f : \Omega \to X^*$ has the RS-property if and only if for every $\epsilon > 0$ there exists $E \in \Sigma$ with $\mu(\Omega \setminus E) < \epsilon$ such that the set $\{(x \circ f)\chi_E : x \in B_X\}$ is weakly pre-compact.*

THEOREM 2.2 [9]. *If a bounded weakly measurable function $f : \Omega \to X^*$ has the RS-property, then there exists a Pettis integrable function $g : \Omega \to X^*$ such that $x \circ f = x \circ g$ in $L_1(\mu)$ for every $x \in X$.*

THEOREM 2.3 [6]. *A bounded linear operator $T : X \to Y$ is weakly compact if and only if $T^* : Y^* \to X^*$ is (ω^*, ω)-continuous.*

THEOREM 2.4 [7]. *A Dunford integrable function $f : \Omega \to X$ is Pettis integrable if and only if the operator $T_f : X^* \to L_1(\mu)$ defined by $T_f(x^*) = x^* \circ f$ is (ω^*, ω)-continuous.*

In particular, if f is Pettis integrable then T_f is a weakly compact operator.

3. Pettis Decomposability

In this section, we investigate some properties of the Pettis decomposable function using the associated operator.

DEFINITION 3.1 [1]. *A function $f : \Omega \to X^*$ is said to be $\mu-$weak* scalarly null if $x \circ f = 0 \quad \mu - a.e.$ for every $x \in X$.*

A bounded weakly measurable function $f : \Omega \to X^*$ is called μ-Pettis decomposable if there exist a Pettis integrable function g and a $\mu -$ weak scalarly null function h such that $f = g + h$.*

THEOREM 3.1. *If X is a separable Banach space, then a bounded weakly measurable function $f : \Omega \to X^*$ is Pettis decomposable if and only if f is Pettis integrable.*

proof. If $f : \Omega \to X^*$ is Pettis integrable, then clearly f is Pettis decomposable.

Conversely, if $f : \Omega \to X^*$ is Pettis decomposable, then there exist a Pettis integrable function g and a $\mu-$weak* scalarly null function h such that $f = g+h$. Let $(x_i)_{i=1}^{\infty}$ be dense in X. Since h is $\mu-$weak* scalarly null, $x \circ h = 0\mu - a.e.$ for every $x \in X$. For each i, let N_i be μ-null sets such that $x_i \circ h(t) = 0$ if $t \in \Omega \setminus N_i$. Then for each i, $x_i \circ h(t) = 0$ if $t \in \Omega \setminus (\cup_{i=1}^{\infty} N_i)$. Let $x \in X$ be arbitrary. Since $(x_i)_{i=1}^{\infty}$ is dense in X, there exists a subsequence $(x_{i_j})_{j=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ such that $\lim_{j\to\infty} x_{i_j} = x$. Hence $x \circ h(t) = (\lim_{j\to\infty} x_{i_j}) \circ h(t) = \lim_{j\to\infty}(x_{i_j} \circ h(t)) = 0$ if $t \in \Omega \setminus (\cup_{i=1}^{\infty} N_i)$. Therefore $h(t) = 0$ if $t \in \Omega \setminus (\cup_{i=1}^{\infty} N_i)$. Since $\mu(\cup_{i=1}^{\infty} N_i) = 0$, $h = 0 \quad \mu - a.e.$ Thus $f = g \quad \mu - a.e.$ Since g is Pettis integrable, f is also Pettis integrable.

THEOREM 3.2. *Let $f : \Omega \to X^*$ be a bounded weakly measurable function. Then $f : \Omega \to X^*$ is Pettis decomposable if and only if the operator $\tilde{T_f}^{**} : X^{**} \to L_1(\mu)$ is (ω^*, ω)-continuous.*

proof. Suppose that $f : \Omega \to X^*$ is Pettis decomposable. Then there exist a Pettis integrable function g and a $\mu -$ weak* scalarly null function h such that $f = g+h$. Since g is Pettis integrable, the operator $T_g : X^{**} \to L_1(\mu)$ defined by $T_g(x^{**}) = x^{**} \circ g$ is (ω^*, ω)-continuous by Theorem 2.4. Hence $\tilde{T_g}^{**} = T_g$ ([4], p685). Since h is $\mu -$ weak* scalarly null, $x \circ f = x \circ g$ in $L_1(\mu)$ for every $x \in X$. Hence $\tilde{T_f} = \tilde{T_g}$, and so $\tilde{T_f}^{**} = \tilde{T_g}^{**}$. Thus $\tilde{T_f}^{**} = T_g$. Therefore $\tilde{T_f}^{**}$ is (ω^*, ω)-continuous.

Conversely, suppose that the operator $\tilde{T_f}^{**} : X^{**} \to L_1(\mu)$ is (ω^*, ω)-continuous. Then $\tilde{T_f}^{*}$ is weakly compact by Theorem 2.3. Hence $\tilde{T_f}$ is also weakly compact, and so $\tilde{T_f}(B_X) = \{\tilde{T_f}(x) : x \in B_X\} = \{x \circ f : x \in B_X\}$ is weakly sequentially compact. Therefore every sequence in $\{x \circ f : x \in B_X\}$ has a weakly convergent subsequence. Hence $\{x \circ f : x \in B_X\}$ is weakly pre-compact. By Theorem 2.1, $f : \Omega \to X^*$ has the RS-property. By Theorem 2.2, there exists a Pettis integrable function $g : \Omega \to X^*$ such that $x \circ f = x \circ g$ in $L_1(\mu)$ for every $x \in X$. Let $h = f - g$, then h is $\mu -$ weak* scalarly null. Therefore $f = g + h$ is Pettis decomposable.

THEOREM 3.3. *Let $f : \Omega \to X^*$ be a bounded weakly measurable function. Then $f : \Omega \to X^*$ is Pettis integrable if and only if $T_f = \tilde{T_f}^{**}$.*

proof. Suppose that $f : \Omega \to X^*$ is Pettis integrable. Then $T_f : X^{**} \to L_1(\mu)$ defined by $T_f(x^{**}) = x^{**} \circ f$ is (ω^*, ω)-continuous by Theorem 2.4. Let $x^{**} \in X^{**}$ be arbitrary. Since X is weak* dense in X^{**}, there exists a net (x_α) in X such that $x_\alpha \xrightarrow{\omega^*} x^{**}$. $T_f(x_\alpha) = \tilde{T_f}(x_\alpha) = \tilde{T_f}^{**}(x_\alpha)$ for each α. Since T_f is (ω^*, ω)-continuous,

$T_f(x_\alpha) \xrightarrow{\omega} T_f(x^{**})$. Since f is Pettis integrable, T_f is weakly compact by Theorem 2.4. Hence $\tilde{T}_f$ is also weakly compact, and so $\tilde{T}_f{}^*$ is weakly compact. By Theorem 2.3, $\tilde{T}_f{}^{**}$ is (ω^*, ω)-continuous. Hence $\tilde{T}_f{}^{**}(x_\alpha) \xrightarrow{\omega} \tilde{T}_f{}^{**}(x^{**})$. Therefore $T_f(x^{**}) = \tilde{T}_f{}^{**}(x^{**})$ for all $x^{**} \in X^{**}$. Thus $T_f = \tilde{T}_f{}^{**}$.

Conversely, suppose that $T_f = \tilde{T}_f{}^{**}$. Then $\tilde{T}_f$ is weakly compact since $\tilde{T}_f{}^{**}(X^{**}) \subseteq L_1(\mu)$. Hence $\tilde{T}_f{}^*$ is also weakly compact, and so $\tilde{T}_f{}^{**}$ is (ω^*, ω)-continuous by Theorem 2.3. By hypothesis, T_f is (ω^*, ω)-continuous. Therefore f is Pettis integrable by Theorem 2.4.

4. Dunford Decomposability

In this section, we introduce the new concept of a Dunford decomposable function which is an extension of Pettis decomposable function and investigate properties of the Dunford decomposable function.

DEFINITION 4.1. *A Dunford integrable function $f : \Omega \longrightarrow X^*$ is called μ-Dunford decomposable if there exist a Dunford equi-integrable function g and a $\mu -$ weak* saclarly null function h such that $f = g + h$.*

LEMMA 4.1. *Let $f : \Omega \longrightarrow X$ be a Dunford integrable function. Then $f : \Omega \to X$ is Dunford equi-integrable if and only if the operator $T_f : X^* \to L_1(\mu)$ defined by $T_f(x^*) = x^* \circ f$ is weakly compact.*

proof. $f : \Omega \to X$ is Dunford equi-integrable if and only if $\{x^* \circ f : x^* \in B_{X^*}$ is uniformly integrable in $L_1(\mu)$. $T_f(B_{X^*}) = \{x^* \circ f : x^* \in B_{X^*}\}$ is uniformly integrable in $L_1(\mu)$ if and only if $T_f(B_{X^*})$ is relatively weakly compact in $L_1(\mu)$. Thus f is Dunford equi-integrable if and only if T_f is weakly compact.

THEOREM 4.2. *If $f : \Omega \longrightarrow X^*$ is Dunford decomposable, then $\tilde{T}_f{}^{**} : X^{**} \to L_1(\mu)^{**}$ is weakly compact.*

proof. If $f : \Omega \longrightarrow X^*$ is Dunford decomposable, then there exist a Dunford equi-integrable function g and a $\mu -$ weak* scalarly null function h such that $f = g + h$. Since h is $\mu -$ weak* scalarly null, $x \circ f = x \circ g$ in $L_1(\mu)$ for every $x \in X$. Hence $\tilde{T}_f = \tilde{T}_g$, and so

$\tilde{T_f}^{**} = \tilde{T_g}^{**}$. Since $g : \Omega \to X^*$ is Dunford equi-integrable, $T_g : X^{**} \to L_1(\mu)$ defined by $T_g(x^{**}) = x^{**} \circ g$ is weakly compact by Lemma 4.1. Hence $\tilde{T_g} : X \to L_1(\mu)$ is weakly compact, and so $\tilde{T_g}^{**}$ is also weakly compact. Therefore $\tilde{T_f}^{**}$ is weakly compact.

THEOREM 4.3. *If $f : \Omega \to X^*$ is Dunford integrable then the following are equivalent:*

(i) *$f : \Omega \to X^*$ is Dunford decomposable.*
(ii) *There exists a Dunford equi-integrable function g such that for every $x \in X$, $T_f(x) = x \circ g$ in $L_1(\mu)$.*
(iii) *For every $\epsilon > 0$, there exist $E \in \Sigma$ and a Dunford equi-integrable function g such that $\mu(\Omega \setminus E) < \epsilon$ and $(x \circ f)\chi_E = x \circ g \quad \mu - a.e.$ for every $x \in X$.*

proof. (i)⇒(ii). If $f : \Omega \to X^*$ is Dunford decomposable, then there exist a Dunford equi-integrable function g and a μ − weak* scalarly null function h such that $f = g + h$. Since h is μ − weak* scalarly null, $x \circ f = x \circ g$ in $L_1(\mu)$ for every $x \in X$. Therefore $T_f(x) = x \circ g$ in $L_1(\mu)$ for every $x \in X$.

(ii)⇒ (iii). Suppose that (ii) holds. Then there exists a Dunford equi-integrable function g such that $x \circ f = x \circ g \quad \mu - a.e.$ for every $x \in X$. Let $E = \Omega$. Then (iii) holds.

(iii)⇒(i). Suppose that (iii) holds. By the exhaustion principle ([5], p70), there exists a Dunford equi-integrable function $g; \Omega \to X^*$ such that $x \circ f = x \circ g \quad \mu - a.e.$ for every $x \in X$. Let $h = f - g$. Then h is μ − weak* scalarly null. Hence $f = g + h$ is Dunford decomposable.

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