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PRIME SERIES

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Let R be a commutative ring with identity and M an R-module. We will use $Spec(R)$ $[Max(R)]$ to denote the prime [maximal] spectrum of R. By a prime series for M we will mean a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where $M_i/M_{i-1} \simeq R/P_i$ for $P_i \in Spec(R)$ $(i = 1, \cdots, n)$. Of course, a prime series is called a composition series if $P_i \in Max(R)$. Observe that if M has a prime series then necessarily M is finitely generated since each M_i/M_{i-1} is cyclic. Given submodules N , N' of M, the set $\{r \in R | \ rN' \subset N\}$ is an ideal of R, which we denote $N : N'$. Similarly, if $I \subset R$ is an ideal then $\{a \in M | aI \subset N\}$ is a submodule of M, which we write $N : I$. The ideal $0 : M$ is called the annihilator of M and written $ann(M)$. For $a \in M$ we write $ann(a) = \{r \in R | \ ra = 0\}$. Note that if I is an ideal in R that is maximal among all annihilators of non-zero elements of R, then I is a prime ideal of R. As usual, we call $P \in Spec(R)$ an associated prime of R if $P = ann(x)$ for some $x \in M$ or equivalently if R/P is isomorphic to a submodule of M. We will use $Ass(M)$ to denote the set (possibly empty) of associated primes of M. The set $\{ \ P \in Spec(R) \mid M_P \neq 0 \ \}$ is called the support of M, and we denote by $Supp(M)$.

If I is an ideal of R, then

$$V(I) \ = \ \{P \in Spec(R) | P \supseteq I\}$$

is called the zero set of I in $Spec(R)$. A subset $S \subseteq Spec(R)$ is called closed if there is an ideal I of R such that $S = V(I)$. For two closed subsets $C_i = V(I_i)$ $(i = 1, 2)$ of $Spec(R)$, $C_1 \cup C_2 = V(I_1 \cap I_2) = V(I_1 I_2)$ is also closed, and for any family $\{C_\lambda\}_{\lambda \in \Lambda}$ of closed subsets the set $\cap_\lambda V(I_\lambda) \ = \ V(\sum_\lambda I_\lambda)$ is also closed in $Spec(R)$. The set $\{V(I) | I$

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is an ideal of $R\}$ is the closed sets of a topology on $Spec(R)$, the Zariski topology on $Spec(R)$. $Max(R)$ will be considered with the subspace topology of the Zariski topology on $Spec(R)$. It is well-known that the Zariski topology is a quasi-compact space but it is rarely Hausdorff space. We say that a nonempty closed set A in a topological space is irreducible if it can not be expressed as a union $A = C_1 \cup C_2$ of two strictly smaller closed sets C_1 and C_2. For any subset $\mathcal{S}$ of $Spec(R)$,

$$\mathcal{T}(\mathcal{S}) = \cap_{P \in \mathcal{S}} P$$

is called the ideal of $\mathcal{S}$ in R.

In the Theorem 1 , we collect the results from [3] , [4] and [6] for the Zariski topology.

THEOREM 1. *Let R be a ring. Then*

(1) *For any subset $\mathcal{S}$ of $Spec(R)$, $V(\mathcal{T}(\mathcal{S})) = \bar{\mathcal{S}}$ is the closure of $\mathcal{S}$ in $Spec(R)$.*

(2) *For any ideal I of R, $\mathcal{T}(V(I)) = Rad(I)$.*

(3) *A closed subset $A \subset Spec(R)$ is irreducible if and only if $\mathcal{T}(A)$ is prime.*

(4) *If R is a Noetherian and M a finitely generated R-module, then $Ass(M)$ is finite and the set of minimal elements of $Ass(M)$ and of $Supp(M)$ coincide. In this case, let $P_1, \cdots, P_n$ be the minimal elements of $Supp(M)$; then*

$$Supp(M) = V(P_1) \cup \cdots \cup V(P_n).$$

In the case of (4), each $V(P_i)$ is a irreducible component of the closed set $Supp(M)$. In this case, $P_1, \cdots, P_n$ are called the isolated associated primes of M, and the remaining associated primes of M are called embedded primes.

For example, let $R = \mathbf{Z}[X]$. One primary decomposition for $(9, 3X + 3)$ is $(3) \cap (9, X+1)$, where (3) is a prime ideal and $(9, X+1)$ is $(3, X+1)$-primary. Thus $Ass((9, 3X+3)) = \{(3), (3, X+1)\}$. So, (3) is an isolated prime ideal and $(3, X+1)$ is an embedded prime ideal of $(9, 3X+3)$.

We observe following properties.

LEMMA 2. *If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is an exact sequence of R-modules, then without any finite assumptions we have*

$$Ass(M') \subseteq Ass(M) \subseteq Ass(M') \cup Ass(M'')$$

and so if an R-module M is the direct sum of a family (M_i) *of submodules then* $Ass(M) = \cup Ass(M_i)$.

Proof. It is clear that $Ass(M') \subseteq Ass(M)$. Let $P \in Ass(M)$. Then M contains a submodule N such that $N \simeq R/P$. If $M' \cap N = \{0\}$, then the image of N is isomorphic to R/P. So $P \in Ass(M'')$. If $M' \cap N \neq \{0\}$, then the annihilator of every nonzero elements in $M' \cap N$ is P. Hence $P \in Ass(M' \cap N) \subseteq Ass(M')$.

We can write $M' \subseteq M$, and $M'' = M/M'$. For any submodule M' of M, we have $Ass(M') \subseteq Ass(M) \subseteq Ass(M') \cup Ass(M/M')$. If R is an integral domain, not a field, then for any prime ideal P,

$$0 \longrightarrow P \longrightarrow R \longrightarrow R/P \longrightarrow 0$$

is exact. Thus $Ass(R) = \{0\}$ and $Ass(R/P) = \{P\}$. Hence the inclusion $Ass(R) \subset Ass(P) \cup Ass(R/P)$ is proper.

Next theorem shows that if M has a prime series, then $Ass(M) \subseteq Supp(M)$.

THEOREM 3. *Let M be an R-module. If M has a prime series*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where,

$$M_i/M_{i-1} \simeq R/P_i \; for \; P_i \in Spec(R) \; (i = 1, \cdots, n),$$

then $Ass(M) \subseteq \{P_1, \cdots, P_n\} \subseteq Supp(M)$.

Proof. Since $Ass(R/P_i) = \{P_i\}$, it follows that

$$Ass(M) \subseteq \{P_1, \cdots, P_n\}.$$

Note that for each i, $(M_i/M_{i-1})_{P_i} \simeq (R/P_i)_{P_i} \simeq R_{P_i}/P_{i P_i}$. Thus $M_{i-1 P_i} \subset M_{i P_i}$ and so $M_{i P_i} \neq \{0\}$ for each i. In particular

$M_{P_i} \neq \{0\}$ and so $P_i \in Supp(M)$. Hence $Ass(M) \subseteq \{P_i, \cdots, P_n\} \subseteq Supp(M)$.

Easy examples show that both containments may be strict. If $R = M = \mathbf{Z}$, then $Ass(\mathbf{Z}) = \{0\}$ and $Supp(\mathbf{Z}) = \{(p) | p = 0$ or p is prime $\}$. Thus $\{p_1, \cdots, p_n\} \subset Supp(\mathbf{Z})$ for any prime series for $\mathbf{Z}$. And since $0 \subset (p_1) \cdots (p_n) \subset (p_2) \cdots (p_n) \subset \cdots \subset (p_n) \subset \mathbf{Z}$ is a prime series with $\{0, (p_1), \cdots, (p_n)\}$ occurring, we may have $Ass(\mathbf{Z}) \subset \{P_1, \cdots, P_n\}$. However, $\mathbf{Z}$ does have a prime series using only 0, namely $0 \subset \mathbf{Z}$.

Let R be a ring and M an R-module. A prime ideal P of R is called a weakly associated if there exists $x \in M$ such that P is a minimal element of the set of prime ideals containing $ann(x)$. We denote the set of all weakly associated primes of M by $Ass_f(M)$.

Note that if P be a prime ideal containing $ann(x)$, then P can be shrunk to a prime ideal minimal among all prime ideals containing $ann(x)$. So $M \neq 0$ if and only if $Ass_f(M) \neq \emptyset$ and $Ass(M) \subset Ass_f(M)$ by the definition.

THEOREM 4. *If R is Noetherian, then $Ass_f(M) = Ass(M)$.*

Proof. Let $P \in Ass_f(M)$. Then there exists $x \in M$ such that P is a minimal prime ideal containing $ann(x)$. Let $N = Rx$. So N is finitely generated with annihilator $ann(Rx) = ann(x)$. Since R is Noetherian, P is the annihilator of a non-zero element of M [2, Theorem 86]. Hence $P \in Ass(M)$. This completes the proof.

A complete lattice is called a multiplicative lattice if there is defined a commutative, associative, join distributive multiplication such that the unit element of the lattice is the identity for the multiplication. Let M be a complete lattice. Recall that M is a lattice module over the multiplicative lattice L, or simply an L-module, if there is a multiplication between elements of L and M, denoted by lD for $l \in L$ and $D \in M$, which satisfies the following properties:

(1) $(lb)D = l(bD)$;
(2) $IB = B$;
(3) $OB = 0_M$;
(4) $(\bigvee_\alpha I_\alpha)(\bigvee_\beta B_\beta) = \bigvee_{\alpha,\beta} I_\alpha B_\beta$;
for all l, b, l_α in L and for all B, B_β in M.

Let M be a L-module. An element A of M is called meet principal [*resp.*join principal] if $(b \bigwedge (B : A))A = bA \bigwedge B$[*resp.*$b \bigvee (B : A) =$

$(bA \bigvee B) : A]$ for all b in L and for all B in M. An element A of M is said to be principal if it is both meet principal and join principal.

An element H in M is called a compact element of M if $H \leq \bigvee_\alpha B_\alpha$ for any set $\{B_\alpha\}_\alpha$ of elements of M implies $H \leq B_{\alpha_1} \bigvee \cdots \bigvee B_{\alpha_n}$ for some subset $\{B_{\alpha_1}, \cdots, B_{\alpha_n}\}$ of $\{B_\alpha\}_\alpha$. If each element of M is a join of principal[compact] elements of M, then M is called a PG-lattice [CG-lattice].

An L-module M is called a K-lattice if it is a CG-lattice and for any compact element $h \in L$ and any compact element $H \in M$, the element hH is compact. An L-module M is called a R-lattice if it is a PG-lattice and every principal element of M is compact. An L-module M is said to be Noetherian if M satisfies the ACC , is modular, and a PG-lattice. If L is a Noetherian L-module, L is called a Noetherian lattice.

For any element b of a multiplicative lattice L, $V(b)$ will denote the set of all prime elements of L containing b.

In [5] Nakker and Anderson generalized $Ass(M)$ and $Ass_f(M)$ and they found nice conditions for which $Ass(M) = Ass_f(M)$.

Let P be a prime element of L. Then we say P is associated [weakly associated] with the module M if there is a non-zero compact element $H \in M$ such that $P = (0 : H)$ [P is minimal in $V((0 : H))$]. The set of prime element associated [weakly associated] with M is denoted by $Ass_L(M)$ [$Ass_{Lf}(M)$] or simply by $Ass(M)$ [$Ass_f(M)$].

In the Theorem 5, we collect the results from [5].

THEOREM 5.

(1) *Let L be a K-lattice and let M be an R-lattice. If M satisfies the ACC, then $Ass(M) = Ass_f(M)$.*

(2) *Let the L-module M be a K-lattice. If L satisfies the ACC, then $Ass(M) = Ass_f(M)$.*

We say that an R-module M is R-flat if for each injective homomorphism $f : N' \to N$ of R-modules, the homomorphism

$$1_M \otimes f : M \otimes_R N' \longrightarrow M \otimes_R N$$

is injective, where 1_M is the identity isomorphism of M.

The following well-known exercise from Bourbaki [1, exercise 17, page 46] will be used to give $Ass_f(R) = Spec(R)$.

PROPOSITION 6. (Bourbaki [1])
The following statements are equivalent:

(1) *Every principal ideal of R is generated by an idempotent.*
(2) *Every R-module is flat.*
(3) $nil(R) = \{0\}$ *and* $Spec(R)$ *is Hausdorff.*

A commutative ring satisfying either of the statements in Proposition 6 is called an absolutely flat ring.

THEOREM 7. *If R is an absolutely flat ring, then $Ass_f(R) = Spec(R)$.*

Proof. Let P be a non-zero prime ideal of R. For $x \in P$, $x \neq 0$, there exists an idempotent e such that $xR = eR$. Thus $e \in P$ and $e(e-1) = 0$. And for any $y \in R$ with $y(e-1) = 0$, we have $y \in P$. Hence $ann(e-1) \subseteq P$.

Suppose that $ann(e-1) \subseteq Q \subset P$ for some prime Q. Then there exists $w \in P - Q$ and $wR = fR$ with $f^2 = f$. Then $w = fr$ for some $r \in R$ and $f \in P$. Since $f(f-1) \in Q$ and $f \notin Q$, we have $f - 1 \in Q \subseteq P$. But this is a contradiction. Thus P is a minimal prime ideal over $ann(e-1)$. Therefore $Ass_f(R) = Spec(R)$.

We say that N is a primary submodule of M if $r \in R$ is a zero-divisor for M/N then $r \in \sqrt{ann(M/N)}$. So a primary ideal is just a primary submodule of the R-module R. Let R be a Noetherian ring and M a finitely generated R-module. Then N is a primary submodule of M if and only if $Ass(M/N)$ has a single element. If $Ass(M/N) = \{P\}$ and $ann(M/N) = I$, then I is primary and $\sqrt{I} = P$.

We next observe that if R is Noetherian and M is a finitely generated R-module, then any $P \in Supp(M)$ occurs in at least one prime series for M.

LEMMA 8. *Let R be Noetherian and let M be a finitely generated R-module. Then any filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_m = M$ of M can be refined to a prime series for M.*

Proof. Each N_i/N_{i-1} is Noetherian and hence M has a prime series.

THEOREM 9. *Let M be a finitely generated R-module where R is Noetherian and $P \in Supp(M)$. Then P occurs in some prime series for M.*

Proof. Let $N = P_P M_P \bigcap M$. Then N is P-primary. Refine $0 \subset N \subset M$ to a prime series for M. Since $Ass(M/N) = \{P\}$, P must occur in this prime series for M.

From the example involving $\mathbf{Z}$ and from Theorem 9, we see that there is no uniqueness involved in the prime series appearing in primes. To get some uniqueness, we make the following definition.

DEFINITION 10. *Let M be a finitely generated R-module where R is Noetherian. Let*

$$Eps(M) = \{P \in Spec(R) | P \text{ appears in every prime series for } M\}.$$

Clearly $Ass(M) \subseteq Eps(M) \subseteq Supp(M)$. Also observe that $Eps(M)$ is a finite set. Example 11 shows that it is possible to have $Eps(M) = Ass(M)$ even though M does not have a composition series using only primes from $Eps(M)$.

EXAMPLE 11. *Let $(X,Y) \subset k[X,Y]$, k a field. Then $Ass((X,Y)) = \{0\}$. No prime series for (X,Y) can use only 0's. For if $A \subset (X,Y)$ with $(X,Y)/A \simeq k[X,Y]/0 = k[X,Y]$, then (X,Y) has $k[X,Y]$ as a proper summand which is absurd. However, since $0 \subset (X) \subset (X,Y)$ and $0 \subset (Y) \subset (X,Y)$ are two prime series for (X,Y) using $\{0,(Y)\}$ and $\{0,(X)\}$, respectively, it follows that $Ass((X,Y)) = \{0\}$, even though no single prime series can use only 0.*

EXAMPLE 12. *Let (R,M) be a one dimensional local domain, not a DVR. Then $Ass(M) = \{0\}$. However, if $I \subset M$ with $M/I \simeq R$, then M has R as a factor, which is a contradiction. Hence any composition series for M must involve M, and so $\{0\} = Ass(M) \subset Esp(M) = \{0, M\} = Supp(M)$.*

A natural question is what modules M have a prime series using only primes from $Ass(M)$ or $Esp(M)$. We will study by considering the case when $M = R$.

DEFINITION 13. *Let R be a Noetherian ring. We call R good if R has a prime series using only primes from $Ass(R)$.*

Of course any domain is good. More generally, a finite direct product of domains is good. Indeed, let $R = R_1 \times \cdots \times R_n$ where each R_i is an integral domain. Let $P_i = R_1 \times \cdots \times R_{i-1} \times 0 \times R_{i+1} \times \cdots \times R_n$. Then $\{P_1, \cdots P_n\}$ is the set of the minimal primes of R and $Ass(R) = \{P_1, \cdots, P_n\}$. Note that

$$P_i \cap \cdots \cap P_n = R_1 \times \cdots \times R_{i-1} \times 0 \times \cdots \times 0$$

and that

$$0 = P_1 \cap \cdots \cap P_n \subset P_2 \cap \cdots \cap P_n \subset \cdots \subsetneq P_n \subset R$$

is a prime series for R (using only $\{P_1, \cdots, P_n\}$) where

$$P_{i+1} \cap \cdots \cap P_n / P_i \cap \cdots \cap P_n \simeq R/P_i$$

for $i = 1, \cdots, n-1$. We will show that if R is a reduced Noetherian ring, then this is the only possible type of prime series.

THEOREM 14. *Let R be a reduced Noetherian ring with n minimal primes. Then R is good if and only if the minimal primes $P_1, \cdots, P_n$ of R can be ordered so that $P_{i+1} \cap \cdots \cap P_n / P_i \cap \cdots \cap P_n$ is cyclic for $i = 1, \cdots, n-1$.*

In this case

$$0 = P_1 \cap \cdots \cap P_n \subset P_2 \cap \cdots \cap P_n$$

$$\subset P_3 \cap \cdots \cap P_n \subset \cdots \subset P_n \subset R$$

is a prime series for R. Moreover, every prime series for R involving on $\{P_1, \cdots, P_n\}$ must have this form.

Proof. ($\Leftarrow$) We will prove this by induction on n. For $n = 1$, R is a domain and the result is trivial. So suppose $n > 1$. Let $(x_1) = P_2 \cap \cdots \cap P_n$ and observe that $(x_1) \simeq R/P_1$. Now $\bar{R} = R/(x_1)$ is a reduced Noetherian ring with minimal primes

$$\bar{P}_2 = P_2/(x_1), \cdots, \bar{P}_n = P_n/(x_1)$$

and

$$\bar{P}_{i+1} \cap \cdots \cap \bar{P}_n / \bar{P}_i \cap \cdots \cap \bar{P}_n \simeq P_{i+1} \cap \cdots \cap P_n / P_i \cap \cdots \cap P_n$$

is cyclic for $i = 2, \cdots, n-1$. By induction $\bar{R} = R/(x_1)$ has a prime series of the form $\bar{0} = \bar{P}_2 \cap \cdots \cap \bar{P}_2 \subset \bar{P}_3 \cap \cdots \cap \bar{P}_n \subset \cdots \subset \bar{P}_n \subset \bar{R}$. Putting the two series together given that

$$0 = P_1 \cap \cdots \cap P_n \subset P_2 \cap \cdots \cap P_n \subset \cdots \subset P_n \subset R$$

is a prime series for R and $P_{i+1} \cap \cdots \cap P_n / P_i \cap \cdots \cap P_n \simeq R/P_i$. Hence R is good.

($\Rightarrow$) Suppose that R is good. Then R has a good prime series

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{m-1} \subset I_m = R\ .$$

Let P be a minimal prime of R, so R_P is a field. By refining $0_P = I_{0_P} \subseteq I_{1_P} \subseteq \cdots \subseteq I_{m_P} = R_P$ to a composition series for R_P, we see that there is exactly one factor I_{i+1}/I_i with $I_{i+1_P}/I_{i_P} \simeq (I_{i+1}/I_i)_P \simeq R_P$ (all the rest are 0), so the factor R/P occurs exactly once in any prime series for R. Hence $m = n$ and by renumbering the primes of R, we can assume that $I_i/I_{i-1} \simeq R/P_i$. In particular, $R/I_{n-1} = I_n/I_{n-1} \simeq R/P_n$ so $I_{n-1} = P_n$. Now $I_{n-2} \subseteq P_n$ and $I_{n-2} \subseteq I_{n-2} : I_{n-1} = P_{n-1}$, so since P_{n-1} and P_n are minimal primes of R, they are minimal over I_{n-2}. Also from the prime series, we get that $Ass(R/I_{n-2}) \subseteq \{P_{n-1}, P_n\}$, so $Ass(R/I_{n-2}) = \{P_{n-1}, P_n\}$. So $I_{n-2} = Q_{n-1} \cap Q_n$ where $Q_{n-1}[Q_n]$ is $P_{n-1}[P_n]$-primary, respectively. Now $P_{n-1} = I_{n-2} : P_n = (Q_{n-1} : P_n) \cap (Q_n : P_n) = Q_{n-1} \cap (Q_n : P_n)$. It follows that $Q_{n-1} = P_{n-1}$ and $Q_n = P_n$, so $I_{n-2} = P_{n-1} \cap P_n$. Continuing in the same manner, we get for each i that $I_i = P_{i+1} \cap \cdots \cap P_n$. Since each I_i/I_{i-1} is cyclic, each $P_{i+1} \cap \cdots \cap P_n / P_i \cap \cdots \cap P_n$ is cyclic.

We get following results.

Corollary 15. *Let R be a reduced Noetherian ring with two minimal primes. Then R is good if and only if at least one minimal prime is principal.*

Corollary 16. *Let R be a reduced Noetherian ring with n minimal primes. If R is good, then some minimal prime of R can be generated by $n-1$ elements.*

Proof. Let $P_1, \cdots, P_n$ be n minimal primes. Then

$$0 = P_1 \cap \cdots \cap P_n \subset P_2 \cap \cdots \cap P_n \subset \cdots \subset P_n \subset R$$

is a good prime series. Now each factor is cyclic, so P_n can be generated by $n-1$ elements.

We next give some conditions equivalent to a ring R having a composition series involving only minimal primes.

THEOREM 17. *Let R be a Noetherian ring with minimal primes $P_1, \cdots$*
, P_n such that for each P_i, R_{P_i} is a SPIR with m_{i1} the least positive integer with $P_{P_i}^{m_{i1}} = 0_{P_i}$ and suppose that $Ass(R) = \{P_1, \cdots, P_n\}$. Then R has a composition series involving only $P_1, \cdots, P_n$ if and only if there exists a sequence of n-tuples of integers $(m_{1k}, \cdots, m_{nk})$ with $k = 1, \cdots, m_{11} + \cdots + m_{n1}$ satisfying

$$0 \leq m_{ij+1} \leq m_{ij} \text{ and } m_{1j+1} + \cdots + m_{nj+1} = m_{1j} + \cdots + m_{nj} - 1$$

so that

$$P_1^{(m_{1j}+1)} \cap \cdot \cdot \cap \ P_n^{(m_{nj}+1)} / P^{(m_{1j})} \ \cap \cdots \cap P^{(m_{nj})}$$

is cyclic.

Proof. $(\Leftarrow)$ Consider the filtration

$$0 = P_1^{(m_{11})} \cap \cdots \cap P^{(m_{n1})} \ \subset \ P_1^{(m_{12})} \cap \cdots \subset \ P_n^{(m_{n2})} \ \subset \ \cdots$$
$$\cdots \subset \ P^{(m_{1l})} \cap \cdots \cap P_n^{(m_{nl})} = R \qquad (**)$$

where $l = m_{11} + \cdots + m_{n1}$. By hypothesis each quotient is cyclic. Also

$$\begin{aligned}[P_1^{(m_{1i})} \cap \cdots \cap P_n^{(m_{ni})}] &: [P_1^{(m_{1i+1})} \cap \cdots \cap P_n^{(m_{ni+1})}] \\ &= \ P_1^{(m_{1i}-m_{1i+1})} \cap \ \cdots \ \cap P_n^{(m_{ni}-m_{ni+1})} \\ &= \ P_j\end{aligned}$$

where j is the unique integer with $m_{ji} - m_{j+1} \ = \ 1$. Thus $(**)$ is good composition series for R.

$(\Rightarrow)$ Suppose that R is good. For each minimal prime P_i, R_{P_i} is a SPIR with index m_{i1}. Since the factor R_{P_i}/P_{iP_i} occurs exactly m_{i1} times in a composition series for R_{P_i}, R/P_i occurs exactly m_{i1} times in any good composition series for R. Thus a good composition series for R has $l \ = \ m_{i1} + \cdots + m_{in}$ terms, say $0 \ = \ I_1 \subset I_2 \subset \cdots \subset I_{l+1} \ = \ R$ is a good composition series for R. Now for each i, $Ass(R/I_i) \subseteq Ass(R)$ so the only associated primes for I_i are $\{P_1, \ \cdots \ P_n\}$, so $I_i \ = \ P_1^{(m_{1i})} \cap \cdots \cap P_n^{(m_{ni})}$. Now $I_k \subseteq I_j \Rightarrow \ I_{kP_i} \subseteq I_{jP_i}$ so $m_{ik} \geq m_{ij}$. Now since the filtration has length l, the sequence $(m_{1k}, \ \cdots \ , m_{nk})$ satisfies the hypothesis of the theorem and hence the quotient is cyclic.

COROLLARY 18. *Let R be a Noetherian Krull domain and let*

$$(X) = P_1^{(m_{11})} \cap \cdots\cdots \cap P_n^{(m_{n1})}$$

be a non-zero nonunit of R where $P_1, \cdots, P_n$ are the height one primes containing X. Then $R/(X)$ has a good composition series if and only if there exists a sequence of integers $(m_{1k}, \cdots, m_{nk})$ as in the previous theorem with each quotient

$$P_1^{(m_{1j+1})} \cap \cdots \cap P_n^{(m_{nj+1})} / P_1^{(m_{1j})} \cap \cdots \cap P_n^{(m_{nj})}$$

cyclic. Hence if $R/(X)$ has a good composition series, one of the minimal primes can be generated by $m_{11} + \cdots + m_{n1}$ elements.

EXAMPLE 19. *Let $R = k[X^2, Y^2, Z^2, XY, XZ, YZ]$, then R is a 3-dimensional Noetherian integrally closed domain. Let*

$$P = (X^2, XY, YZ) = (X)k[X, Y, Z] \cap R.$$

Now $X^2Z^2 = (XZ)^2 \in P^2$, so $X^2Z^2 \in P^{(2)}$ and $Z^2 \notin P$, thus $X^2 \in P^{(2)}$. Hence $P^{(2)} = (X^2)$. Now $P/P^{(2)}$ in $R/P^{(2)}$ is not principal, so $R/P^{(2)}$ is not good.

THEOREM 20. *Let R be a Noetherian ring. Then there is an ideal I of R having a composition series using exactly the associate primes of R.*

Proof. Let $\mathcal{S} = \{J \neq 0$ an ideal of $R \mid J$ has a composition series using only primes from $Ass(R)\}$. Now for $P \in Ass(R)$, let $P = ann(R_{x_1})$, since $R_{x_1} \in \mathcal{S}$, $\mathcal{S} \neq \phi$. Let I be a maximal element of $\mathcal{S}$. If $I = R$, we are done. So assume $I \neq R$. If $Ass(R/I) \cap Ass(R) \neq \emptyset$, then we can extend I. So we can assume that $Ass(R/I) \cap Ass(R) = \emptyset$. Now $Ass(R) \subseteq Ass(I) \cup Ass(R/I)$, so $Ass(R) = Ass(I)$. But since all primes of $Ass(I)$ are usual in the composition series for I, we have the result.

More generally, an almost identical proof gives the next theorem.

THEOREM 21. *If R is a Noetherian ring and M a finitely generated R-module. Then there is a submodule N of M having a composition series using exactly the associated primes of M. Moreover, $Ass(M/N) \cap Ass(M) = \emptyset$.*

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