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CONDITIONAL EXPECTATIONS OF FUZZY RANDOM SETS

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1. Introduction

The concept of a fuzzy random variable was introduced by Puri and Ralescu[4] as a model to represent relationships between the outcomes of a random experiment and inexact data. Also, Puri and Ralescu[5] introduced the conditional expectation of a fuzzy random variable by using Radon-Nikodym theorem for fuzzy set-valued measures. Stojakovic[6] has studied the fuzzy conditional expectation slightly different than that of Puri and Ralescu[5]. But these are the corresponding results of fuzzy set-valued functions with compact support which are based on the theory of multivalued function established by Hiai and Umegaki[2].
The purpose of this paper is to define conditional expectation of a fuzzy random variable in more general setting than that of Puri and Ralescu[5] and Stojakovic[6]. The terminology "fuzzy random set" is used instead of fuzzy random variable since it is a natural generalization of random set. Section 2 consists of the basic facts about integrablly bounded random sets. In section 3, we study the space of integrablly bounded fuzzy random sets. Finally, in section 4 we introduce the conditional expectation of a fuzzy random set and examine its properties.

2. Preliminaries

Throughout this paper, let (Ω, Σ, P) be a complete probability space and Λ a real separable Banach space with norm $\| \ \|$ and let $L(\Sigma) = L(\Omega, \Sigma, P, \Lambda)$ denote the Banach space of measurable functions $f : \Omega \to \Lambda$ such that

$$\|f\| = \int_\Omega \|f(\omega)\| dP \qquad \text{is finite.}$$

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Let $\mathcal{K}(\Lambda)$ denote the family of all nonempty, closed subsets of Λ. Moreover, we denote by $\mathcal{K}_c(\Lambda)$ the family of all convex $A \in \mathcal{K}(\Lambda)$, and by $\mathcal{K}_{cc}(\Lambda)$ the family of all compact convex $A \in \mathcal{K}(\Lambda)$. A linear structure in $\mathcal{K}(\Lambda)$ is defined via the operations

$$A \oplus B = cl\,(A + B),$$
$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in \mathcal{K}(\lambda)$ and $\lambda \in R$.
For $A, B \in \mathcal{K}(\Lambda)$, the number $\delta(A, B) \geq 0$ is defined by

$$\delta(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|a - b\|\}$$

Note that if A and B are bounded, then $\delta(A, B)$ is the Hausdorff metric of A and B.
A random set is defined as a measurable multivalued function $F : \Omega \to \mathcal{K}(\Lambda)$ which its graph $G(F) = \{(\omega, x) \in \Omega \times \Lambda : x \in F(\omega)\}$ is $\Sigma \times \mathcal{B}$ - measurable, where $\mathcal{B}$ is the Borel σ-field of Λ . A random set F is called integrably bounded if there exists a integrable function $g : \Omega \to R$ such that $\sup_{x\in F(\omega)} \|x\| \leq g(\omega)$ for all $\omega \in \Omega$.
Let $\mathcal{L}(\Omega, \Sigma, P; \Lambda) = \mathcal{L}(\Sigma)$ denote the space of all integrably bounded random sets, where $F, G \in \mathcal{L}(\Sigma)$ are considered to be identical if $F(\omega) = G(\omega)\, a.s.$. For $F \in \mathcal{L}(\Sigma)$, $S_F = \{f \in L(\Sigma) : f(\omega) \in F(\omega)\, a.s.\}$ denotes the measurable selection of F.
Now, for $F, G \in \mathcal{L}(\Sigma)$, we define

$$\Delta(F, G) = \int_\Omega \delta(F(\omega), G(\omega))dP$$

Then Δ is a metric on $\mathcal{L}(\Sigma)$. Also, if we define

$$\mathcal{L}_c(\Omega, \Sigma, P; \Lambda) = \mathcal{L}_c(\Sigma) = \{F \in \mathcal{L}(\Sigma) : F(\omega) \in \mathcal{K}_c(\Lambda)\, a.s.\}$$

$$\mathcal{L}_{cc}(\Omega, \Sigma, P; \Lambda) = \mathcal{L}_{cc}(\Sigma) = \{F \in \mathcal{L}(\Sigma) : F(\omega) \in \mathcal{K}_{cc}(\Lambda)\, a.s.\}$$

then $\mathcal{L}(\Sigma)$ is a complete metric space with respect to the metric Δ and $\mathcal{L}_c(\Sigma) \supset \mathcal{L}_{cc}(\Sigma)$ are closed subspaces of $\mathcal{L}(\Sigma)$.(see Theorem 3.3 of [2])

3. Spaces of Integrablly Bounded Fuzzy Random Sets

For a fuzzy set u in Λ, the α-level set of u is denoted by $L_\alpha u = \{x \in \Lambda : u(x) \geq \alpha\}$ where $0 \leq \alpha \leq 1$. A fuzzy set u is called convex (closed) if $L_\alpha u$ is convex (closed) set for each $\alpha \in [0,1]$, and also is called compact if $L_\alpha u$ is compact set for each $\alpha \in (0,1]$. The convex hull of a fuzzy set u in Λ is defined by

$$(co\, u)(x) = \inf\{v(x) : v \text{ is convex and } v(x) \geq u(x) \text{ for all } x \in \Lambda\}$$

Then $co\, u$ is a convex fuzzy set and $L_\alpha(co\, u) = co\,(L_\alpha u)$ for each $\alpha \in [0,1]$. Also, the closure of u is defined by

$$(cl\, u)(x) = \inf\{v(x) : v \text{ is closed and } v(x) \geq u(x) \text{ for all } x \in \Lambda\}.$$

Then $cl\, u$ is a closed set and $L_\alpha(cl\, u) = cl\,(L_\alpha u)$ for each $\alpha \in [0,1]$. The closed convex hull $cl\,(co\, u)$ of u is denoted by $\overline{co}\, u$.
An extension of $\mathcal{K}(\Lambda)$ is obtained by defining the space $\mathcal{F}(\Lambda)$ of all closed fuzzy sets u with the property $\{x \in \Lambda : u(x) = 1\} \neq \emptyset$. We denote by $\mathcal{F}_c(\Lambda)$ the family of all convex $u \in \mathcal{F}(\Lambda)$, and by $\mathcal{F}_{cc}(\Lambda)$ the family of all compact convex $u \in \mathcal{F}(\Lambda)$. For $u, v \in \mathcal{F}(\Lambda)$ and $\lambda \in R$, we define

$$d(u,v) = \sup_{\alpha > 0} \delta(L_\alpha u, L_\alpha v)$$
$$(u \oplus v)(x) = \sup\{\alpha \in [0,1] : x \in L_\alpha u \oplus L_\alpha v\}$$
$$(\lambda u)(x) = \begin{cases} u(x/\lambda), & \text{if } \lambda \neq 0 \\ \chi_{\{0\}}, & \text{if } \lambda = 0 \end{cases}$$

Lemma 3.1. The spaces $\mathcal{F}(\Lambda), \mathcal{F}_c(\Lambda)$ and $\mathcal{F}_{cc}(\Lambda)$ are closed under addition $\oplus$ and scalar multiplication.

Proof. $L_\alpha(\lambda u) = \lambda L_\alpha u$ is trivial for every $\alpha \in [0,1]$. Thus, it suffices to prove that $L_\alpha(u \oplus v) = L_\alpha u \oplus L_\alpha v$ for every $\alpha \in [0,1]$. In order to prove this, we use lemma 1 of [6]. Let $M_\alpha = L_\alpha u \oplus L_\alpha v$. Then it follows immediately that $M_0 = \Omega$ and $M_\beta \subset M_\alpha$ if $\alpha \leq \beta$.
Now let $\alpha_1 \leq \alpha_2 \leq \ldots, \lim \alpha_n = \alpha > 0$. Then $\bigcap_{n=1}^{\infty} M_{\alpha_n} \supset M_\alpha$.
Using the property of δ (Debreu [1], p362),

$$\delta(M_{\alpha_n}, M_\alpha) \leq \delta(L_{\alpha_n} u, L_\alpha u) + \delta(L_{\alpha_n} v, L_\alpha v) \to 0$$

as $n \to \infty$. Thus, $\bigcap_{n=1}^{\infty} M_{\alpha_n} = M_\alpha$.
Hence lemma 1 of [6] is applicable, and we obtain that

$$L_\alpha(u \oplus v) = M_\alpha = L_\alpha u \oplus L_\alpha v$$

for each $\alpha \in [0,1]$. □

A fuzzy random set is defined as a function $X : \Omega \to \mathcal{F}(\Lambda)$ such that the function $L_\alpha X : \Omega \to \mathcal{K}(\Lambda)$ defined by

$$L_\alpha X(\omega) = \{x \in \Lambda : X(\omega)(x) \geq \alpha\}$$

is a random set for all $\alpha \in [0,1]$. A fuzzy random set X is called integrably bounded if $L_\alpha X$ is integrably bounded for each $\alpha \in (0,1]$. Let $\Phi(\Omega, \Sigma, P; \Lambda) = \Phi(\Sigma)$ be the family of al' integrably bounded fuzzy random sets. Two fuzzy random sets $X, Y \in \Phi(\Sigma)$ are considered to be identical if $L_\alpha X = L_\alpha Y \, a.s.$ for all $\alpha \in [0,1]$. Moreover, we denote by $\Phi_c(\Sigma)$ the family of all integrably bounded fuzzy random set $X : \Omega \to \mathcal{F}_c(\Lambda)$, and by $\Phi_{cc}(\Sigma)$ the family of all integrably bounded fuzzy random set $X : \Omega \to \mathcal{F}_{cc}(\Lambda)$.
For $X, Y \in \Phi(\Sigma)$, we define $D(X,Y) = \sup_{\alpha>0} \Delta(L_\alpha X, L_\alpha Y)$. Then $\Phi(\Sigma)$ is a complete metric space with respect to the metric D and $\Phi_c(\Sigma) \supset \Phi_{cc}(\Sigma)$ are closed subspaces of $\Phi(\Sigma)$.
We now define the following three operations in the family $\Phi(\Sigma)$.
(1) Addition: for $X, Y \in \Phi(\Sigma)$,

$$(X \oplus Y)(\omega) = X(\omega) \oplus X(\omega), \ \omega \in \Sigma.$$

(2) Multiplication by measurable functions: for $X \in \Phi(\Sigma)$ and a measurable real-valued function ξ ,

$$(\xi X)(\omega) = \xi(\omega) X(\omega), \ \omega \in \Omega.$$

(3) Closed convex hull: for $X \in \Phi(\Sigma)$,

$$(\overline{co}\, X)(\omega) = \overline{co}\, X(\omega)$$

Theorem 3.2. (1) $\Phi(\Sigma), \Phi_c(\Sigma)$ and $\Phi_{cc}(\Sigma)$ are closed under addition $\oplus$ and multiplication by L^∞-functions.

(2) The mapping $X \mapsto \overline{co}\,X$ is a nonexpansive mapping from $\Phi(\Sigma)$ to $\Phi_c(\Sigma)$, i.e.,

$$D(\overline{co}\,X, \overline{co}\,Y) \leq D(X,Y) \qquad \text{for } X, Y \in \Phi(\Sigma).$$

(3) $D(X_1 \oplus X_2, Y_1 \oplus Y_2) \leq D(X_1, Y_1) + D(X_2, Y_2)$ for $X_i, Y_i \in \Phi(\Sigma), i = 1, 2$.

Proof. (1) It follows immediately from lemma 3.1.
(2) From theorem 3.5 of Hiai and Umegaki [2],

$$\begin{aligned}\Delta(L_\alpha(\overline{co}\,X), L_\alpha(\overline{co}\,Y) &= \Delta(\overline{co}\,(L_\alpha X), \overline{co}\,(L_\alpha Y)) \\ &\leq \Delta(L_\alpha X, L_\alpha Y)\end{aligned}$$

for each $\alpha > 0$. This implies $D(\overline{co}\,X, \overline{co}\,Y) \leq D(X,Y)$.
(3) can be proved by the equality

$$\begin{aligned}\Delta(L_\alpha(X_1 \oplus X_2), L_\alpha(Y_1 \oplus Y_2)) &= \Delta(L_\alpha X_1 \oplus L_\alpha X_2, L_\alpha Y_1 \oplus L_\alpha Y_2) \\ &\leq \Delta(L_\alpha X_1, L_\alpha Y_1) + \Delta(L_\alpha X_2, L_\alpha Y_2)\end{aligned}$$

□

4. Conditional Expectations of Fuzzy Random Sets

Throughout this section, $\mathcal{F}$ denotes a sub-σ-field of Σ. For $F \in \mathcal{L}(\mathcal{F})$, we define

$$S_F(\mathcal{F}) = \{f \in L(\mathcal{F}) : f(\omega) \in F(\omega)\, a.s.\}.$$

If $F \in \mathcal{L}(\mathcal{F})$,then there exists a unique $E(F|\mathcal{F}) \in \mathcal{L}(\mathcal{F})$ such that

$$S_{E(F|\mathcal{F})}(\mathcal{F}) = cl\,\{E(f|\mathcal{F}) : f \in S_F\}$$

where the closure is taken with repect to the norm in $L(\Sigma)$.
$E(F|\mathcal{F}) \in \mathcal{L}(\mathcal{F})$ is called the conditional expectation of F relative to $\mathcal{F}$.

Lemma 4.1. If $\{F_n\} \subset \mathcal{L}(\Sigma)$ and $F_n \overset{\Delta}{\rightarrow} F$, then $E(F_n|\mathcal{F}) \overset{\Delta}{\rightarrow} E(F|\mathcal{F})$.

Proof. It follows immediately from that the mapping $F \mapsto E(F|\mathcal{F})$ is a nonexpansive mapping from $\mathcal{L}(\Sigma)$ to $\mathcal{L}(\mathcal{F})$. □

Corollary 4.2. If $\{F_n\} \subset \mathcal{L}(\Sigma)$ is a decreasing sequence such that

$$\bigcap_{n=1}^{\infty} F_n(\omega) = F(\omega)$$

for all $\omega \in \Omega$, then

$$\bigcap_{n=1}^{\infty} E(F_n|\mathcal{F}) = E(F|\mathcal{F}).$$

Proof. First we note that

$$E(F_1|\mathcal{F}) \supset E(F_2|\mathcal{F}) \supset \cdots \supset E(F|\mathcal{F}).$$

Since $F_n \xrightarrow{\Delta} F$, we have $E(F_n|\mathcal{F}) \xrightarrow{\Delta} E(F|\mathcal{F})$. Hence,

$$E(F|\mathcal{F}) = \bigcap_{n=1}^{\infty} E(F_n|\mathcal{F})$$

. □

Theorem 4.3. If $X \in \Phi(\Sigma)$, then there exists a unique fuzzy random set $Y \in \Phi(\mathcal{F})$ such that

$$L_\alpha Y = E(L_\alpha X|\mathcal{F})$$

for all $\alpha \in [0,1]$.

Proof. Let $F_\alpha = E(L_\alpha X|\mathcal{F})$. Then F_α is $\mathcal{F}$-measurable random set. Since $L_0 X(\omega) = \Lambda$ for all $\omega \in \Omega$, it follows that $S_{L_0 X} = L$ and hence $F_0(\omega) = \Lambda$ for all $\omega \in \Omega$. Trivially, $F_\beta \subset F_\alpha$ if $\alpha \leq \beta$.
Now let $\alpha_1 \leq \alpha_2 \leq \ldots, \lim \alpha_n = \alpha > 0$. Then

$$L_{\alpha_1} X(\omega) \supset L_{\alpha_2} X(\omega) \supset \cdots \supset L_\alpha X(\omega)$$

and

$$L_\alpha X(\omega) = \bigcap_{n=1}^{\infty} L_{\alpha_n} X(\omega)$$

for all $\omega \in \Omega$. By corollary 4.2,

$$\begin{aligned} F_\alpha = E(L_\alpha X|\mathcal{F}) &= \bigcap_{n=1}^{\infty} E(L_{\alpha_n} X|\mathcal{F}) \\ &= \bigcap_{n=1}^{\infty} F_{\alpha_n} \end{aligned}$$

An application of lemma 1 in [6] shows that there exists $Y \in \Phi(\mathcal{F})$ such that

$$L_\alpha Y = F_\alpha = E(L_\alpha X|\mathcal{F})$$

for all $\alpha \in [0,1]$. The uniqueness of Y is trivial. □

The fuzzy random set $Y \in \Phi(\mathcal{F})$ defined above is called conditional expectation of $X \in \Phi(\Sigma)$ relative to $\mathcal{F}$. We use the notation $Y = E(X|\mathcal{F})$.

Theorem 4.4. The conditional expectation $E(X|\mathcal{F})$ of $X \in \Phi(\Sigma)$ have the following properties:
(1) If $X \in \Phi_c(\Sigma)$, then $E(X|\mathcal{F}) \in \Phi_c(\mathcal{F})$.
(2) $D(E[X|\mathcal{F}]), E[Y|\mathcal{F}]) \leq D(X, Y)$ for all $X, Y \in \Phi(\Sigma)$.
(3) $E(X \oplus Y|\mathcal{F}) = E(X|\mathcal{F}) \oplus E(Y|\mathcal{F})$ for all $X, Y \in \Phi(\Sigma)$.
(4) $E(\xi X|\mathcal{F}) = \xi E(X|\mathcal{F})$ for $X \in \Phi(\Sigma)$ and $\mathcal{F}$-measurable function $\xi \in L^\infty$.
(5) $E(\overline{co}\, X|\mathcal{F}) = \overline{co}\, E(X|\mathcal{F})$ for $X \in \Phi(\Sigma)$.

Proof. (1) If $X \in \Phi_c(\Sigma)$, then $L_\alpha X \in \mathcal{L}_c(\Sigma)$ for all $\alpha \in [0,1]$. By theorem 5.1 of [2],

$$L_\alpha E(X|\mathcal{F}) = E(L_\alpha X|\mathcal{F}) \in \mathcal{L}_c(\mathcal{F})$$

which implies $E(X|\mathcal{F}) \in \Phi_c(\mathcal{F})$.
(2) Using the fact that the mapping $F \mapsto E(F|\mathcal{F})$ is a nonexpansive

mapping from $\mathcal{L}(\Sigma)$ to $\mathcal{L}(\mathcal{F})$ (theorem 5.2 of [2]), we have

$$\begin{aligned} D(E[X|\mathcal{F}], E[Y|\mathcal{F}]) &= \sup_{\alpha\geq 0}\Delta(L_\alpha E[X|\mathcal{F}], L_\alpha E[Y|\mathcal{F}]) \\ &= \sup_{\alpha\geq 0}\Delta(E[L_\alpha X|\mathcal{F}], E[L_\alpha Y|\mathcal{F}]) \\ &\leq \sup_{\alpha\geq 0}\Delta(L_\alpha X, L_\alpha Y) \\ &= D(X,Y) \end{aligned}$$

The proofs of (3),(4) and (5) can be proceeded in a similar arguments. □

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