Kangweon–Kyungki Math. Jour. 2 (1994), pp. 65–72

A STUDY OF PRIME SUBMODULES AND MAXIMAL SUBMODULES

Soung–chul Lee and Heang–lai Lee

1. Introduction

In this note, R is a commutative ring with identity and M is a unital R-module.

A proper submodule N of M is said to be **prime** or **p-prime** (resp. **primary** or **p-primary**) if $re \in N$ for $r \in R$ and $e \in M$ implies that either $e \in N$ or $r \in p = (N : M) = \{r \in R | rM \subseteq N\} = Ann_R(M/N)$ (resp. for if $re \in N$ for $r \in R$ and $e \in M$ implies that either $e \in N$ or $r \in p = \sqrt{(N : M)} = \{r \in R | r^n M \subseteq N$ for some $n \in Z^+\})$, where p is a prime ideal of a ring R. In other words, N is a prime submodule if it is a primary submodule whose radical is identical with $p = (N : M))$. An R-module M is called **torsion-free** if $rm \neq 0$ for all non-zero divisors $r \in R$ and $0 \neq m \in M$. Clearly, every prime ideal of a ring R is a prime submodule of the R-module R and every prime submodule of an R- module is primary.

An R-module M is called a **multiplication module** if, for every submodule N, there exists an ideal I of R such that $N = IM$. It is well-known that M is a multiplication module if and only if $N = (N : M)M$ for every submodule N of M.

Following [8], we shall call a submodule N of an R-module M a maximal p-primary submodule of M if N is a p-primary submodule which is not strictly contained by any other p-primary submodules of M. Maximal p–prime submodule of M can be defined in a similar way.

Let S be a multiplicatively closed set in R and let T be the set of all ordered (x, S) where $x \in M$ and $s \in S$. Define a relation on T by $(x_1, s_1) \backsim (x_2, s_2)$ if there exists $t \in S$ such that $(x_1 s_2 - x_2 s_1)t = 0$. This is an equivalence relation on T, and we denote the equivalence class of (x, s) by $[x/s]$. Let M_s denote the set of equivalence classes of T with respect to this relation. Clearly, M_s is the R_s-module. Since R

Received January 18, 1994

may be considered as an R-module we can form the quotient module R_s. Clearly, R_s is a ring.

2. Prime submodules of an R-module

PROPOSITION 2-1. *If a proper submodule N of an R-module M is a prime submodule, then $(N : M)$ is a prime ideal of R.*

Proof. It follows from [4, p.41, Proposition 2-11].

PROPOSITION 2-2. *Let N be a proper submodule of an R- module M with $(N : M) = p$. Then the following statements are equivalent:*

(1) N is a prime submodule of M.

(2) M/N is a torsion-free R/p-module.

(3) $(N : (r)) = N$ for every $r \in R - p$.

(4) $(N : A) = N$ for every ideal $A \not\subseteq p$.

(5) $(N : (e)) = p$ for every $e \in M - N$.

(6) $(N : L) = p$, i.e., $Ann(L/N) = Ann(M/N)$ for every submodule L of M properly containing N.

Proof. (1) $\Longleftrightarrow$ (2) : ($\Rightarrow$) Let $Rm + N \in M/N$, $m + N \neq N$, and let $r + p \in Ann(m + N)$, then $N = (r + p)(m + N)$, so $rm \in N$. Since $m \notin N$ and N is a prime submodule, $r \in Ann(M/N) = p$ and hence $Ann(m + N) = p$. Thus M/N is a torsion-free R/p-module.

($\Leftarrow$) Let $re \in N$, $e \in N$, then $(r + p)(e + N) = N$. Since M/N is a torsion-free R/p-module, $r + p \in Ann(M/N) = p$, which implies that $r \in p$. Thus N is a prime submodule.

(2)$\Rightarrow$(3) : Clearly, $N \subseteq (N : (r))$ for every $r \in R$. Let $e \in (N : (r))$ for every $r \in R - p$. Then $re \in N$ and so $(r + p)(e + N) = N$. Since M/N is a torsion-free R/p-module and $r \notin p = Ann(M/N)$, $e+N = N$ and so $e \in N$. Thus $(N : (r)) = N$ for every $r \in R - p$.

(3)$\Rightarrow$(4) : Let A be an ideal of R with $A \not\subseteq p$ and let $r \in R-p$. Then $(N : A) \subseteq (N : (r)) = N$, and hence $(N : A) = N$ since $N \subseteq (N : A)$.

(4)$\Rightarrow$(2) : Let $e + N$ be a torsion-element of M/N and $r \in A - p$. Then $(r + p)(e + N) = N$, and so $e \in N$ (by (4)). Hence M/N is a torsion-free R/p-module.

(1)$\Rightarrow$ (5) : Let $e \in M$. Then $(N : (e)) \subseteq (N : M) = p$. Let $r \in (M : (e))$ with $e \notin N$. Then $re \in N$. Since N is a prime submodule and $e \notin N$, $r \in (N : M) = p$. Hence $(N : (e)) = p$ for every $e \in M - N$.

(5)$\Rightarrow$(6) : Let L be a submodule of M properly containing N. Then there exists $e \in L - N$ and so $p = (N : M) \supseteq (N : L)$. Since $p = (N : M) \subseteq (N : L)$, $p = (N : L)$.

(6)$\Rightarrow$(1) : Let $re \in N$ with $e \in M - N$ and let $L = Re + N$. Then L is a submodule of M properly containing N and $rL \subseteq N$. Hence $p = (N : L)$ and $r \in (N : L) = p$. Thus N is a prime submodule.

REMARKS. *Let N be a submodule of an R-module M. Then the following hold*

(i) if $N = M$, then N is prime ;

(ii) if $RM \subseteq N$, then N is prime since $R \subseteq (N : M)$.

PROPOSITION 2-3. *If N is a submodule of an R-module M that $(N : M)$ is a maximal ideal of R, then N is a prime submodule. In particular, mM is a prime submodule of an R-module M for every maximal ideal m of R such that $mM \neq M$.*

Proof. Since $(N : M) = p$ is a maximal ideal, M/N is a vector space over the field R/p, and so M/N is a torsion-free R/p-module. Hence N is prime by Proposition 2-1.

PROPOSION 2-4. *Let N be a proper submodule of an R-module M and let m be a maximal ideal of R. Then N is m-prime if and only if $mM \subseteq N$. Consequently, if N is an m-prime submodule of M, then so is every proper submodule of M containing N.*

PROPOSITION 2-5. *If N is a maximal submodule of an R- module M, then N is a prime submodule and $(N : M)$ is a maximal ideal of R.*

Proof. N is a maximal submodule of M if and only if M/N is a simple R-module. Hence M/N is a cyclic R-module $(m + N)R$ with a generator $m + N$ and $Ann_R(m + N) = Ann_R(M/N) = (N : M)$ is a maximal ideal of R. It follows that N is a prime from Proposition 2-3.

COROLLARY 2-6. *If M is a finitely generated module, then every proper submodule of M is contained in a prime submodule.*

Proof. It follows from [3. p.28, Theorem.2.3.11].

PROPOSITION 2-7. *Let $N_1, N_2, \cdots, N_k$ be submodules of an R-module M and let N be a prime submodule of M. If $N_1 \cap N_2 \cap \cdots \cap N_k \subseteq N$, then there exists an i such that either $N_i \subseteq N$ or $(N_i : M) \subseteq (N : M)$.*

Proof. Assume the contrary, then there exists an $e \in N_1$ such that $e \notin N$ and $p_i \in (N_i : M)$ such that $p_i \notin (N : M)$ for every $i \neq 1$. Thus $p_i e \in (N_1 : N_i)$ for every $i \neq 1$ so that $p_2 p_3 \cdots p_k e \in N_1 \cap N_2 \cap \cdots \cap N_k \subseteq N$. However, $e \notin N$ and $p_2 p_3 \cdots p_k e \notin (N : M)$, which contradicts that N is prime.

THEOREM 2-8. *Let M be a multiplication module. If N is a prime submodule of M, then there exists a unique prime ideal p of R containing $Ann(M)$ such that $N = pM$.*

Proof. Since M is a multiplication module and N is a prime submodule of M, $N = (N : M)M = (pM : M) = p$ for some prime ideal p of R with $Ann(M) \subseteq p$. We show that $(N : M) = (pM : M) = p$ for the uniqueness. Clearly $p \subseteq (pM : M)$. If $r \in (pM : M)$, then $(r)M \subseteq pM$ and so $(r) \subseteq p$ or $(p : (r))M = M$ by [5, Lemma 2.2]. Suppose $(r) \nsubseteq p$, then $(p : (r))M = M$. Clearly $p \subseteq (p : (r))$. If $a \in (p : (r))$, then $a(r) \subseteq p$ and so $ar \in p$. Since $r \notin p$ and p is prime ideal of R, $a \in p$. Thus $(p : (r)) \subseteq p$. Hence $(p : (r)) = p$ and $M = (p : (r))M = pM$. It contradicts to $pM \notin M$. Thus $r \in p$ and $(pM : M) = p$.

COROLLARY 2-9. *Let M be a torsion-free multiplication module and let p be a prime ideal of R. Then pM is prime submodule if and only if $(pM : M) = p$.*

Proof. Since M is torsion-free, $Ann(M) \subseteq p$. If pM is a prime submodule of M, then $pM = (pM : M)M$ and $(pM : M)$ is a prime ideal of R. By Theorem 2-3, $p = (pM : M)$. Conversely, if $p = (pM : M)$, then $pM \neq M$. By [2, Corollary 2-11] it holds.

COROLLARY 2-10. *Let M be a multiplication module and let p be a maximal ideal of R. Then pM is a maximal submodule of M if and only if $(pM : M) = p$.*

Proof. Since $pM \neq M$, $Ann(M) \subseteq p$. It is obvious by Theorem 2-8 and [2, Theorem 2.5].

3. Prime submodules and maximal submodules

Let I be an ideal of a ring R. The **radical** of I, designated by $\sqrt{I}$, is the set $\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n \in Z^+\}$. In particular, an ideal I of the ring R is said to be a **radical ideal** if $\sqrt{I} = I$. One of the most fundamental properties of radical ideals of a ring is that $\sqrt{IJ} = \sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}$ [1, Theorem 5-10].

PROPOSITION 3-1. *Let R be a principal ideal domain. A nontrivial ideal (a) of R is prime if and only if it is a maximal ideal.*

Proof. Suppose that $a + \sqrt{I}$ is a nilpotent element of $R/\sqrt{I}$. Then there exists some $n \in Z^+$ such that $(a + \sqrt{I})^n = a^n + \sqrt{I} = \sqrt{I}$; that is to say, the element $a^n \in \sqrt{I}$. Hence, $(a^n)^m = a^{nm} \in I$ for some positive integer m. This implies that $a \in \sqrt{I}$ and, consequently , that $a + \sqrt{I} = \sqrt{I}$, the zero element of $R/\sqrt{I}$.

As regards the converse, assume that R/I has no nonzero nilpotent elements and let $a \in \sqrt{I}$. Then, for some positive integer $n, a^n \in I$. Passing to the quotient ring R/I, this simply means that $(a + I)^n = I$; in other words, $a + I$ is nilpotent in R/I. By supposition, we must have $a + I = I$ and, in consequence, the element $a \in I$. Our argument shows that $\sqrt{I} \subseteq I$; since the other inclusion always holds, $I = \sqrt{I}$, so that I is radical.

PROPOSITION 3-2. *N is a maximal p-primary submodule of a module if and only if N is a maximal prime submodule of M.*

Proof. Assume that N is a maximal p-primary submodule of M. If $r \notin (N : M)$ then $(N : (r))$ is p-primary submodule by [8, p.100, Proposition 21]. Thus $(N : (r)) = N$ for every $r \notin (N : M)$ by the maximal property of N, whence N is a prime submodule by Proposition 2-2. Since $(N : M)$ is a prime ideal, $(N : M) = \sqrt{(N : M)} = p$, which means that N is a p-prime submodule. Now that N is a maximal p-prime submodule is easy to see. The converse can be verified by using Proposition 2-1.

PROPOSITION 3-3. *Let M be a finitely generated module and N a p-primary submodule. Then the following statements are equivalent :*

(1) N is a maximal p-primary submodule of M ;

(2) N is a maximal p-prime submodule of M ;

(3) for each submodule L of M satisfying $N \subset L \subseteq M$, we have $(N : L) = p$ and $(L : M) \supset p$ (strict inclusion).

Proof. In view of Proposition 3-2, it suffices to prove only the equivalence of (2) and (3). To prove (3)⇒(2), let N be a submodule of M satisfying the two conditions (a) $(N : L) = p$ and (b) $(L : M) \supset p$ for every submodule L of M such taht $N \subset L \subseteq M$. Due to Proposition 2-2, condition (a) is equivalent to that N is p-prime. Condition (b) implies that every submodule L containing N is not p-prime. Thus N is a maximal p-prime submodule of M, so (3) implies (2). Conversely, we assume that N is a maximal p-prime submodule of M and let L be a submodule of M which contains N properly. Then, clearly $(N : M) \supseteq p$ and we have $(N : L) = p$ from Proposition 2-2. Applying [8, p.160, Theorem 10] and Proposition 3-2, we can see that N_p is a maximal pR_p-prime submodule of the R_p-module M_p. Now assume that $(L : M) = p$. Then since M is a finitely generated R-module, $L_p \neq M_p$ by virtue of [8, p.158, Proposition 13]. It follows that L_p is a proper submodule of M_p which contains the pR_p-prime submodule N_p, therefore L_p is a pR_p-prime submodule by Proposition 2-4. By the maximality of N_p we have $N_p = L_p$, whence $N = N_p \cap M = L_p \cap M \supseteq L$, a contradiction. Thus $(L : M) \supset p$, so (2) implies (3).

REMARK. *If m is a maximal ideal of a ring R, then not every m-prime submodule of an R-module M is a maximal submodule. For example, (0) is a maximal ideal of any field F and all maximal or non-maximal subspaces of a vector space V over F are (0)-prime submodules in V.*

THEOREM 3-4. *Let R be a principal ideal domain and let M be a multiplication R-module. A proper submodule N of M is prime if and only if it is a maximal submodule of M.*

Proof. Suppose that N is a prime submodule of M, and let L be a submodule such that $N \subset L \subseteq M$. Then by Proposition 2-1 $(N : M) = p$ is a prime ideal of R, and so p is a maximal ideal of R from Proposition 3-1. Since $Ann(M) = (0 : M) \subseteq (N : M) = p$, $(pM : M) = p$ by Theorem 2-8. By Corollary 2-10, N is a maximal submodule of M. Conversely, assume that N is a maximal submodule of M. Then N is a prime submodule of M by Proposition 2-5.

COROLLARY 3-5. *Let R be a principal ideal domain and let M be a multiplication R-module. A prime submodule N of M is a maximal p-prime submodule.*

Proof. Suppose that N is a p-prime submodule of M. Then by Theorem 3-4 N is a maximal submodule. Hence by Proposition 3-3 N is a maximal p-prime submodule.

COROLLARY 3-6. *Let N be a principal ideal domain and let M be a multiplication R-module. Then the following statements are equivalent*

(1) N is a prime submodule of M ;
(2) N is a maximal submodule of M ;
(3) N is a maximal p-primary submodule of M ;
(4) N is a maximal p-prime submodule of M ;
(5) for each submodule L of M satisfying $N \subset L \subseteq M$, we have $(N : L) = p$ and $(L : M) \supset p$ (strict inclusion).

Proof. It follows from Proposition 3-3, Theorem 3-4 and Corollary 3-5.

References

[1] D.M. Burton, *A first course in Rings and ideals*, Addison-Wesley Publishing Company, 1970.
[2] Z.A. El-Bast and P.F. Simith, *Multiplication modules*, Comm. in Algebra **16** (1988), 655–779.
[3] F. Kasch, *Modules and rings*, Academic Press, 1982.
[4] M.D. Larsen and P.J. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, 1971.
[5] D.S. Lee and H.B. Lee, *Some Remarks on Faithful Multiplication Modules*, Journal of the CHUNGCHENG Math. Society. **6** (1993).
[6] S.C. Lee, *M-radical of submodules in a multiplication module.*, Dan-Kook Univ (1994).
[7] C.P. Lu, *Prime submodules of modules*, Comment. Math. Univ. Sancti Pauli **33, No.1** (1984), 61–69.
[8] D.G. Northcott, *Lessons on rings, modules, and multiplicities*, Cambridge University Press, 1968.

Soung–chul Lee
Department of Mathematics

Samchok National University
Samchok 245-080, Korea

Heang–lai Lee
Department of Mathematics
Dankook University
Seoul 140-210, Korea