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THE CARTESIAN PRODUCTS ON EXTENDED JIANG SUBGROUP

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1. Introduction.

We [6] introduced an extended Jiang subgroup $J(f, x_0, G)$ of the fundamental group of a transformation group as a generalization of the Jiang subgroup $J(f, x_0)$ and gave a necessary and sufficient condition for $J(f, x_0, G)$ to be isomorphic to $J(f, x_0) \times G$.

F. Rhodes defined a path $\theta(\alpha_x, \alpha_y)$ in $X \times Y$ from (x_1, y_1) to (x_2, y_2) for a path α_x in X from x_1 to x_2 and a path α_y in Y from y_1 to y_2. He showed that the mapping $\theta_* : \sigma(X, x_0, G) \times \sigma(Y, y_0, H) \longrightarrow \sigma(X \times Y, x_0 \times y_0, G \times H)$ by $\theta_*([\alpha_x : g], [\alpha_y : h]) = [\theta(\alpha_x, \alpha_y) : (g, h)]$ is an isomorphism in [3].

In this paper, we consider about $J(f \times k, x_0 \times y_0, G \times H)$ if $f \times k$ is a self map from the Cartesian product space $X \times Y$ to itself. We show that the mapping

$$\theta_* : J(f, x_0, G) \times J(k, y_0, H) \longrightarrow J(f \times k, x_0 \times y_0, G \times H)$$

is an isomorphism and the direct product $J(f, x_0) \times J(k, y_0)$ is isomorphic to the Jiang's subgroup $J(f \times k, x_0 \times y_0)$.

2. Preliminaries and main results.

Every pair of homeomorphisms g in G and h in H gives rise to a homeomorphism (g, h) from the product space $Z = X \times Y$ to itself given by $(g, h)(x, y) = (gx, hy)$. A path α_x in X from x_1 to x_2 and a path α_y in Y from y_1 to y_2 give rise to a path $\theta(\alpha_x, \alpha_y)$ in $X \times Y$ from (x_1, y_1) to (x_2, y_2) defined by

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$$\theta(\alpha_x, \alpha_y) = \begin{cases} (\alpha_x(2t), y_1), & 0 \le t \le 1/2 \\ (x_2, \alpha_y(2t-1)), & 1/2 \le t \le 1. \end{cases}$$

Clearly, $(g,h)\theta(\alpha_x, \alpha_y) = \theta(g\alpha_x, h\alpha_y)$ and $\theta(\alpha_x\rho, \alpha_y\rho) = \theta(\alpha_x, \alpha_y)\rho$.

LEMMA 1. *[3] Let α_x and α'_x be paths in X from x_1 to x_2 and from x_2 to x_3. Let α_y and α'_y be paths in Y from y_1 to y_2 and from y_2 to y_3. Then $\theta(\alpha_x + \alpha'_x, \alpha_y + \alpha'_y)$ is homotopic to $\theta(\alpha_x, \alpha_y) + \theta(\alpha'_x, \alpha'_y)$.*

Let $f : X \longrightarrow X$ be a self map and $k : Y \longrightarrow Y$ be a self map. Then $\theta(\alpha_x, \alpha_y)$ is a path of order (g,h) with base point $z_0 = (f(x_0), k(y_0))$ and the homotopy class of $\theta(\alpha_x, \alpha_y)$ depends only on the homotopy classes of α_x and α_y.

THEOREM 2. *Let (X, G) and (Y, H) be transformation groups and let f and k be self maps of X and Y respectively, where X and Y are pathwise connected CW-complexes. Then $J(f, x_0, G) \times J(k, y_0, H)$ is isomorphic to $J(f \times k, x_0 \times y_0, G \times H)$.*

Proof. We know that $(X \times Y, G \times H)$ is a transformation group such that $(g,h)(x,y) = (gx, hy)$. Let

$$\theta(\alpha_x, \alpha_y) = \begin{cases} (\alpha_x(2t), k(y_0)), & 0 \le t \le 1/2 \\ (gf(x_0), \alpha_y(2t-1)), & 1/2 \le t \le 1 \end{cases}$$

for $\quad [\alpha_x : g] \in J(f, x_0, G), \quad [\alpha_y : h] \in J(k, y_0, H)$.

Define $\theta_* : J(f, x_0, G) \times J(k, y_0, H) \longrightarrow J(f \times k, x_0 \times y_0, G \times H)$ by

$$\theta_*([\alpha_x : g], [\alpha_g : h]) = [\theta(\alpha_x, \alpha_y) : (g,h)].$$

Since $[\alpha_x : g]$ be an element of $J(f, x_0, G)$ and $[\alpha_y : h]$ be an element of $J(k, y_0, H)$, there exist a homotopy $H : X \times I \longrightarrow X$ and a homotopy $K : Y \times I \longrightarrow Y$ such that $H(x,0) = f(x)$, $H(x,1) = gf(x)$, $H(x_0, t) = \alpha_x(t)$ and $K(y,0) = k(y)$, $K(y,1) = hk(y)$, $K(y_0, t) = \alpha_y(t)$.

Therefore, there exists a homotopy $W : X \times Y \times I \longrightarrow X \times Y$ such that

$$W(x,y,t) = \begin{cases} (H(x,2t), k(y)), & 0 \leq t \leq 1/2 \\ (gf(x), K(y, 2(t-1/2))), & 1/2 \leq t \leq 1. \end{cases}$$

This homotopy satisfies

$$W(x,y,0) = (H(x,0), k(y)) = (f(x), k(y)),$$
$$W(x,y,1) = (gf(x), K(y,1)) = (gf(x), hk(y)) = (g,h)(f(x), k(y))$$

and $W(x_0, y_0, t) = \theta(\alpha_x, \alpha_y)$. Thus $[\theta(\alpha_x, \alpha_y) : (g,h)]$ belongs to $J(f \times k, x_0 \times y_0, G \times H)$. Since the homotopy class of $\theta(\alpha_x, \alpha_y)$ depends only on the homotopy classes of α_x and α_y, θ_* is well defined.

Let $[\alpha_x : g], [\alpha'_x : g']$ be elements of $J(f, x_0, G)$ and let $[\alpha_y : h], [\alpha'_y : h']$ be elements of $J(k, y_0, H)$. We show that

$$\begin{aligned} &\theta_*(([\alpha_x : g], [\alpha_y : h]) * ([\alpha'_x : g'], [\alpha'_y : h'])) \\ &= \theta_*([\alpha_x + g\alpha'_x : gg'], [\alpha_y + h\alpha'_y : hh']) \\ &= [\theta(\alpha_x + g\alpha'_x, \alpha_y + h\alpha'_y) : (gg', hh')] \end{aligned}$$

and

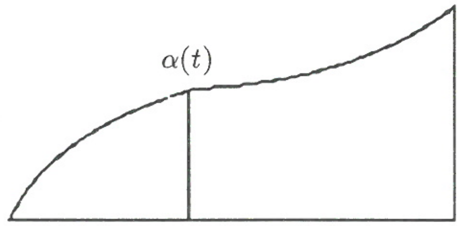
$$\begin{aligned} &\theta_*([\alpha_x : g], [\alpha_y : h]) * \theta_*([\alpha'_x : g'], [\alpha'_y : h']) \\ &= [\theta(\alpha_x, \alpha_y) : (g,h)] * [\theta(\alpha'_x, \alpha'_y) : (g', h')] \\ &= [\theta(\alpha_x, \alpha_y) + (g,h)\theta(\alpha'_x, \alpha'_y) : (g,h)(g',h')] \\ &= [\theta(\alpha_x, \alpha_y) + \theta(g\alpha'_x, h\alpha'_y) : (gg', hh')]. \end{aligned}$$

So, θ_* is a homomorphism by Lemma 5-1.

Let $\phi_x : X \times Y \longrightarrow X$ and $\phi_y : X \times Y \longrightarrow Y$ be projections and $[\alpha : (g,h)]$ be any element of $J(f \times k, x_0 \times y_0, G \times H)$. There exists a $(f \times k)$-homotopy $W : X \times Y \times I \longrightarrow X \times Y$ such that $W(x,y,0) =$

$(f \times k)(x,y)$, $W(x,y,1) = (g,h)(f \times k)(x,y)$ and $W(x_0, y_0, t) = \alpha(t)$. Since there exists a homotopy

$$H(t,s) = \begin{cases} (\phi_x \alpha(3t), k(y_0)), & 0 \le t \le s/3 \\ (\phi_x \alpha(s), \phi_y \alpha(3t-s)), & s/3 \le t \le 2s/3 \\ \alpha(3(1-s)(t-2s/3)/(3-2s)+s), & 2s/3 \le t \le 1, \end{cases}$$



α is homotopic to $\theta(\phi_x \alpha, \phi_y \alpha)$. Let W_{y_0} be a continuous map from $X \times I$ to $X \times Y$ such that $W_{y_0}(x,t) = W(x, y_0, t)$ and let W_{x_0} be a continuous map from $Y \times I$ to $X \times Y$ such that $W_{x_0}(y,t) = W(x_0, y, t)$. Then there exists a f-homotopy $\phi_x \circ W_{y_0} : X \times I \longrightarrow X$ such that

$$\phi_x \circ W_{y_0}(x,0) = f(x),\ \phi_x \circ W_{y_0}(x,1) = gf(x),\ \phi_x \circ W_{y_0}(x_0,t) = \phi_x \alpha(t)$$

and there exists a k-homotopy $\phi_y \circ W_{x_0} : Y \times I \longrightarrow Y$ such that

$$\phi_y \circ W_{x_0}(y,0) = k(y),\ \phi_y \circ W_{x_0}(y,1) = hk(y),\ \phi_y \circ W_{x_0}(y_0,t) = \phi_y \alpha(t).$$

Therefore $[\phi_x \alpha : g]$, $[\phi_y \alpha : h]$ belong to $J(f, x_0, G)$, $J(k, y_0, H)$ respectively and $\theta_*([\phi_x \alpha : g], [\phi_y \alpha : h]) = [\theta(\phi_x \alpha, \phi_y \alpha) : (g,h)]$. Hence θ_* is onto.
Thus θ_* is an isomorphism.

COROLLARY 3. *Let* $(X,G),(Y,H)$ *be transformation groups and* X,Y *be pathwise connected* CW*-complexes. Then the direct product of Jiang subgroups* $J(f,x_0)\times J(k,y_0)$ *is isomorphic to the Jiang subgroup* $J(f\times k, x_0\times y_0)$.

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