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# AUTOMORPHIC FORMS ON ORTHOGONAL GROUPS ATTACHED TO QUADRATIC EXTENSIONS

YOUNG HO PARK

## §1. NOTATIONS

Let $F$ be a number field, and $E/F$ be a quadratic extension of $F$.

We choose a $\beta \in E$ such that $E = F(\beta)$, and let $q(x) = x^2 - \beta^2$ be the irreducible polynomial of $\beta$ over $F$. Then the Galois group of $E/F$ is generated by the automorphism $\sigma$ of order 2, where

$$(a + b\beta)^\sigma = a - b\beta, \text{ for } a, b \in F.$$

Let $Tr(x) = x + x^\sigma$, $N(x) = xx^\sigma$ be the usual trace and norm of $E/F$. Define a bilinear form ( , ) on $E$ by

$$(x, y) = \frac{1}{2} Tr(xy^\sigma).$$

Note that $(x, x) = N(x)$. Let $O$ be the isometry group of $E$,( , ), viewed as an algebraic group defined over $F$. We also let

$$E^1 = \{x \in E \mid N(x) = 1\}.$$

For each place $v$ of $F$, let $F_v$ be the completion of $F$ at $v$, and let $E_v = F_v \otimes_F E$. Then $E_v = F_v[\bar{\beta}]$, where $1, \bar{\beta}$ are linearly independent over $F_v$ and $q(\bar{\beta}) = 0$. Define $\sigma_v : E_v \to E_v$ by $(a + b\bar{\beta})^\sigma = a - b\bar{\beta}$. Via the usual imbedding $E \hookrightarrow E_v$, $a + b\beta \mapsto a + b\bar{\beta}$, $\sigma_v$ is an automorphism of $E_v$ of order 2 extending $\sigma$.

If $q$ is irreducible over $F_v$ (i.e. $v$ is *inert.*), then $E_v = F_v(\beta)$ is a field. If $q$ is reducible over $F_v$ (i.e. $v$ *splits*), then $E_v \simeq F_v \oplus F_v$, via $a + b\bar{\beta} \mapsto (a + b\beta, a - b\beta)$. Note that $F_v$ is imbedded into $F_v \oplus F_v \simeq E_v$ diagonally, and $\sigma_v$ on $F_v \oplus F_v$ is given by $(a, b)^\sigma = (b, a)$.

Let $Tr_v$, and $N_v$ be the trace and norm on $E_v/F_v$. Then $(x, y)_v = \frac{1}{2} Tr_v(xy^\sigma)$ and $O_v = O(E_v)$ is the isometry group of $E_v$,( , )$_v$. Note that when $v$ splits, $(x, y)_v = (xy, xy) \in F_v$, for $(x, y) \in F_v \oplus F_v$.

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## §2. Structure of $O$.

For any anisotropic $y \in E$, let $\tau_y$ be the symmetry defined by

$$\tau_y(x) = x - \frac{2(x,y)}{(y,y)} y.$$

Then $O$ is generated by $\tau_y$, where $y$ runs over the anisotropic vectors [O]. We have

$$\tau_y(x) = -\frac{y}{y^\sigma} x^\sigma.$$

For $u \in E^1$, let $m_u : E \to E$ be the automorphism defined by $m_u(x) = ux$. Then $m_u \in O$, and $\sigma m_u \sigma^{-1} = m_{u^\sigma}$. By the Hilbert theorem 90, we see that $O$ is generated by $m_u (u \in E^1)$, and $\sigma$. In other words,

PROPOSITION 2.1. $O(E) \simeq E^1 \rtimes \langle\sigma\rangle$ □

Note that $O_v \simeq E_v^1 \rtimes \langle\sigma\rangle$, and that if $v$ splits, so that $E_v = F_v \oplus F_v$, then $E_v^1 = \{(x,y) \mid xy = 1\} \simeq F_v^\times$.

For a finite place $v$ of $F$, let $\mathcal{O}_v$, $\mathcal{U}_v$ be the ring of integers, ring of units of $F_v$, respectively. Then $L_v = \mathcal{O}_v + \beta\mathcal{O}_v$ is a maximal compact subring of $E_v$. Let.

$$K_v = \begin{cases} (E_v^1 \cap L_v) \rtimes \langle\sigma_v\rangle & \text{if } v \text{ is inert,} \\ \mathcal{U}_v \rtimes \langle\sigma_v\rangle & \text{if } v \text{ splits.} \end{cases}$$

Then $K_v$ is a maximal compact subgroup of $O_v$. Now suppose $v$ is an archimedean place. We have

$$O_v = \begin{cases} \mathbb{R}^\times \rtimes \langle\sigma_v\rangle & \text{if } v \text{ is real and splits,} \\ \mathbb{C}^1 \rtimes \langle\sigma_v\rangle \simeq O(2) & \text{if } v \text{ is real and inert,} \\ \mathbb{C}^\times \rtimes \langle\sigma_v\rangle & \text{if } v \text{ is complex.} \end{cases}$$

So we let

$$K_v = \begin{cases} \{\pm 1\} \rtimes \langle\sigma_v\rangle & \text{if } v \text{ is real and splits} \\ O(2) & \text{if } v \text{ is real and inert} \\ \mathbb{C}^1 \rtimes \langle\sigma_v\rangle \simeq O(2) & \text{if } v \text{ is complex} \end{cases}$$

and $K_\infty = \prod_{v \text{ archimedean}} K_v$, $K = \prod_v K_v$.

Let $\mathbb{A}$ denote the adele ring of $F$ as usual. Then $O(\mathbb{A}) = \prod_v O_v$, restricted direct product with respect to $K_v$. Since $(\ ,\ )$ is anisotropic over $F$, we see that $O(F)\backslash O(\mathbb{A})$ and $E^1(F)\backslash E^1(\mathbb{A})$ are compact.

## §3. Automorphic representations of $O$.

Since $O_v$ is a semidirect product of commutative $E_v^1$ and finite $\langle\sigma_v\rangle$, we may apply a Mackey's result to obtain the following [L]

PROPOSITION 3.1. *Let $\tau_v$ be an irreducible representation of $O_v$. Then there is an associated character $\chi_v$ of $E_v^1$ such that $\tau_v$ is one of the following representations:*

(1) *$\mathrm{Ind}_{E_v^1}^{O_v}\chi_v$, where $\chi_v^2 \neq 1$,*
(2) *$\chi_v \otimes 1$, where $\chi_v^2 = 1$ and $1$ is the trivial representation of $\langle\sigma_v\rangle$,*
(3) *$\chi \otimes \det$, where $\chi_v^2 = 1$ and $\det$ is the nontrivial representation of $\langle\sigma_v\rangle$.* □

To be more precise, let $f_1, f_2 \in \tau_v = \mathrm{Ind}_{E_v^1}^{O_v}\chi_v$ be defined by

$$f_1(g) = \begin{cases} \chi(x) & \text{if } g = x \in E_v^1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_2(g) = \begin{cases} \chi(x) & \text{if } g = x\sigma_v \in E_v^1\sigma_v, \\ 0 & \text{otherwise,} \end{cases}$$

Then $\tau_v$ is generated by $f_1$ and $f_2$ and $\tau_v(\sigma_v)f_1 = f_2$. Thus $f_1 + f_2$ is fixed by $\sigma_v$, while $f_1 - f_2$ changes the sign. It is not difficult to show that $\mathrm{Ind}_{E_v^1}^{O_v}\chi_v$ are irreducible if and only if $\chi_v^2 = 1$, and that if $\chi^2 = 1$, then $\tau_v = (\chi_v \otimes 1) \oplus (\chi_v \otimes \det)$, with the basis $f_1 + f_2$ and $f_1 - f_2$, respectively.

DEFINITION 3.2. *Suppose $v$ is a finite pace of $F$, which splits. We say that a character $\chi_v$ of $E_v^1$ is unramified if it is trivial on $E_v^1 \cap L_v$.*

Recall that a representation $\tau_v$ of $O_v$ is unramified if it contains a $K_v$-fixed vector. Thus $\tau_v$ is unramified if and only if its associated character $\chi_v$ is unramified, and $\tau_v \neq \chi_v \otimes \det$.

Now let $\tau$ be an admissible irreducible representation of $O(\mathbb{A})$. Then $\tau = \otimes_v \tau_v$, where $\tau_v$ is irreducible representation of $O_v$ such that almost all $\tau_v$ are unramified [F]. We recall the Langlands characterization of automorphic representations of a reductive group $G$ [La].

PROPOSITION 3.3. *A representation $\tau$ of $G(\mathbb{A})$ is an automorphic representation if and only if $\tau$ is a constituent of $\mathrm{Ind}_{P(\mathbb{A})}^{O(\mathbb{A})}\lambda$ for some parabolic subgroup $P$ of $G$ with Levi factor $M$ and some cuspidal representation $\lambda = \otimes_v \lambda_v$ of $M(\mathbb{A})$.* $\square$

In our case, the only nontrivial parabolic subgroup of $O$ is $E^1$. Note that the Levi factor of $E^1$ is itself. Since the constituents of Ind $\lambda$ are the representations $\tau = \otimes_v \tau_v$, where $\tau_v$ is a constituent of $\mathrm{Ind}_{E_v^1}^{O_v} \lambda_v$ and almost all $\lambda_v$ are unramified, we obtain

THEOREM 3.4. *$\tau = \otimes \tau_v$ is an irreducible automorphic representation of $O(\mathbb{A})$ if and only if there is a character $\chi = \otimes \chi_v$ of $E^1(F)\backslash E^1(\mathbb{A})$ such that $\tau_v$ is one of the 3 types given in Proposition 3.1 with associated character $\chi_v$ and the number of $v$'s such that $\tau_v = \chi_v \otimes \det$ is even.* $\square$

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Department of Mathematics
Yonsei University
Seoul, 120-749, Korea