Kangweon–Kyungki Math. Jour. **3** (1995), No. 1, pp. 5–10

STABLE RANKS AND REAL RANKS OF C^*–ALGEBRAS II

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1. Introduction and notations.

Let A be a C^*-algebra. We shall denote the Murray-von Neumann equivalence of two projections p and q in A by $p \sim q$ (i.e. there exists a partial isometry v in A such that $v^*v = p$ and $vv^* = q$). $p \precsim q$ means that p is equivalent to a subprojection of q.

A C^*-subalgebra B of A is said to be *hereditary* if $0 \le a \le b \in B$ and $a \in A$ implies $a \in B$. An equivalent condition is that $B = L \cap L^*$, where L is a left closed ideal of A. For positive $x \in A$, we denote A_x the hereditary C^*-subalgebra of A generated by x.

Stable rank(real rank, resp.) of A denoted by $sr(A)(RR(A)$, resp.) is defined. See [3] and [6] for the exact definition. Note that if $sr(A) < \infty$ then A is stably finite. And $sr(A) = 1(RR(A) = 0$, resp.) is equivalent to the fact that the set of invertible(self-adjoint invertible, resp.) elements of A is dense in $A(A_{sa}$, resp).

A is said *purely infinite* if every nonzero hereditary C^*-subalgebra of A contains an infinite projection. This is equivalent for A to have real rank zero and every nonzero projection of A is infinite ([9]). If A is simple and has a tracial state, then A must be stably finite. If A is not stably finite, then there are three possibilities: (1) A is finite but not stably finite; (2) A is infinite and there is a finite projection; (3) Every nonzero projection of A is infinite. All known examples of infinite simple unital C^*-algebras are purely infinite.

In this note, we find conditions for infinite simple C^*-algebras to be purely infinite.

Received June 8, 1994.
This work was supported by GARC-KOSEF,1993.

2. Purely infinite simple C^*-algebras.

Recall that an element x of a C^*-algebra A is said *well-supported* if there is a projection $p \in A$ with $x = xp$ and x^*x is invertible in pAp. If x is well-supported, then the hereditary C^*-subalgebra of A generated by x is unital.

LEMMA 1. *If a C^*-algebra A has a sequence of mutually orthogonal nonzero projections, then there is a positive element which is not well-supported.*

Proof. We first modify the proof of theorem 6 in [5]. Assume that every positive element in A is well-supported. Take a sequence of mutually orthogonal projections $q_n, n \in \mathbf{N}$. Then $y = \sum_{n=1}^{\infty}(2^n)^{-1}q_n$ is a positive element of A, so there exists a projection $q \in A$ such that $yq = y$ and y^*y is invertible in qAq. Since

$$\frac{1}{2^n}q_n = q_n y = q_n(yq) = \frac{1}{2^n}q_n q$$

we have $q_n \leq q$ for each $n = 1, 2, 3, \cdots$. Now consider $z_m = y^*y - (2^{2m})^{-1}q$ for $m = 1, 2, 3, \cdots$. Then

$$\begin{aligned} z_m &= \sum_{n=1}^{\infty} \frac{1}{2^{2n}}q_n - \frac{1}{2^{2m}}(q_m + (q - q_m)) \\ &= \sum_{n \neq m}^{\infty} \frac{1}{2^{2n}}q_n - \frac{1}{2^{2m}}(q - q_m). \end{aligned}$$

So $q_m z_m = z_m q_m = 0$ for $m = 1, 2, 3, \cdots$. Hence z_m is not invertible in qAq, and the spectrum of y^*y in qAq contains the sequence $\{(2^{2m})^{-1} : m = 1, 2, 3, \cdots\}$. Therefore the spectrum contains 0, which contradicts to the fact that y is well-supported by q. Hence not every positive element of A is well-supported.

COROLLARY 2. *If A is a unital purely infinite simple C^*-algebra , then there is a hereditary C^*-subalgebra B such that $sr(B) < sr(A)$.*

Proof. Let x be a nonzero positive element which is not well supported. Since A_y is either unital or stable for every nonzero positive $y \in A$ by [9], we have that A_x is stable. Since $sr(A_x) = 1$ or 2 by [6] and $sr(A) = \infty$, the proof is completed.

PROPOSITION 3. *Let A be a σ-unital simple C^*-algebra with real rank zero such that A_x is infinite dimensional for every $x \in A^+$. Then either A is purely infinite or there are infinitely many mutually orthogonal finite projections.*

Proof. Suppose that there are only finitely many mutually orthogonal finite projections in A. We shall show that A_x contains an infinite projection for every nonzero positive $x \in A$. First assume that $x \in A$ is a non-well-supported positive element. Since $RR(A) = 0$, A_x has a fundamental approximate identity, i.e. there is an approximate identity $\{e_n\}$ consisting of projections such that $e_{n+1} - e_n \precsim e_n - e_{n-1}$ for all $n \in N$ by corollary 1.3 of [10]. Suppose that every projection of A_x is finite. Since $(e_{n+1} - e_n)(e_n - e_{n-1}) = 0$, there is a $k \in N$ such that $e_k = e_{k+1} = \cdots$. This shows that A_x is unital. This is a contradiction to the fact that x is not well-supported. Hence some projection of A_x is infinite. Secondly let $x \in A$ be well-supported. Then A_x is a unital simple C^*-algebra with $RR(A) = 0$. Hence any nonzero hereditary C^*-subalgebra of A_x contains a nonzero projection. By 4.1 of [1], A_x contains a sequence of mutually orthogonal nonzero projections. By Lemma 1, there is a non well-supported element $y \in A_x$. Hence by the first part of this proof, $A_y \subset A_x$ has an infinite projection. Therefore for every nonzero positive $x \in A$, A_x has an infinite projection . Hence A is purely infinite, completing the proof.

PROPOSITION 4. *Let A be a unital simple C^*-algebra such that every projection is infinite and the annihilator of every $x \in A_{sa}$ has an infinite projection. Then A is purely infinite.*

Proof. It suffices to show that $RR(A) = 0$. Let $x \in A_{sa}$ and B be the annihilator of x. Then there is an infinite projection p in B. By [4], there is a partial isometry v in A such that $vv^* = 1 - p$ and $v^*v = q \leq p$. Set $u = v + v^* + (p - q)$. Then u is a self-adjoint unitary with $uqu = 1 - p$. For $\epsilon > 0$, take $y = x + \epsilon u$. Since

$$y^2 = x^2 + \epsilon^2 + \epsilon(xu + ux) = x^2 + \epsilon^2 + \epsilon(xv + v^*x)$$

write

$$y^2 = \begin{pmatrix} \epsilon^2 p & \epsilon v^* x \\ \epsilon x v & x^2 + \epsilon^2(1-p) \end{pmatrix}.$$

To show that y^2 is invertible, compute

$$x^2 + \epsilon^2(1-p) - \epsilon x v(\epsilon^{-2}p)\epsilon v^* x = x^2 + \epsilon^2(1-p) - x(1-p)x = \epsilon^2(1-p).$$

Since this element is invertible in $(1-p)A(1-p)$, y^2 is invertible by [3] and hence y is invertible. Since $\|x - y\| = \epsilon$, $RR(A) = 0$. This completes the proof.

PROPOSITION 5. *Let A be a unital infinite simple C^*-algebra such that $RR(A) = 0$ and for every nonzero partial isometry v with $v^2 = 0$, vv^* is an infinite projection. Then A is purely infinite.*

Proof. It suffices to show that every nonzero projection is infinite. Since A is infinite, there are mutually orthogonal infinite projections p_1, p_2 that are equivalent to 1. Since $p_2 \leq (1-p_1)$, there is a projection p such that p and $1-p$ are both infinite. Let q be a nonzero finite projection. Then by the standard argument, we have that

$$q \sim pxp \sim (1-p)y(1-p)$$

for some $x, y \in A$. Then there exists a partial isometry $v \in A$ such that

$$v^*v = pxp,$$
$$vv^* = (1-p)y(1-p).$$

Hence

$$q \sim vv^* \geq vpv^* \sim pv^*vp = v^*v.$$

Since vv^* is finite, $(1-p)y(1-p) = vv^* = vpv^*$. Hence $pvp = 0$ and $(vp)^2 = 0$. Note that since $v = vpxp, v(1-p) = 0$, and hence $v = vp$ This shows that $v^2 = 0$. This shows that 0 is the only finite projection. This completes the proof.

It is well-known that for a unital C^*-algebra A, the convex hull of unitaries is dense in the closed unit ball A_1 of A. If the convex hull of extreme points of A_1 is equal to A_1, then A is said to be *extremally rich ([2])*.

PROPOSITION 6. *A unital simple C^*-algebra A is extremally rich if and only if either A is purely infinite or $sr(A) = 1$.*

Proof. Suppose that A is extremally rich and not purely infinite. Then there exists a hereditary C^*-subalgebra B of A such that every projection of B is finite. Since $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$, we may assume that every projection of A is finite. Since A is simple, every extreme point of A_1 is either an isometry or a coisometry and hence a unitary. Therefore A_1 is the convex hull of unitaries. Thus $sr(A) = 1$ by [8]. The converse was shown in [7] and [8].

COROLLARY 7. *For any unital simple C^*-algebra A, $A \otimes B$, where B is a UHF-algebra, is extremally rich.*

COROLLARY 8. *Let A be a unital simple C^*-algebra such that for any zero divisor x, there exists a unitary $u \in A$ such that $ux \geq 0$. Then either A has stable rank 1 or is purely infinite.*

Proof. Let x be a zero divisor and u a unitary in A such that $ux = p \geq 0$. Then $x = u^*p \in u^*\overline{GL(A)} = \overline{GL(A)}$. Hence by [7], the convex hull of $V(A) = \{v \mid v \text{ is either an isometry or a coisometry }\}$ is equal to A_1. Hence by Proposition 6, either $sr(A) = 1$ or A is purely infinite.

COROLLARY 9. *If A is a finite AW^*-algebra, then $sr(A) = 1$.*

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