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ON THE SEMIHOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS

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§1.Introduction

Hörmander[1], Scheurer[3] obtained two kinds of theorems called the uniqueness theorems and the existence theorems from which we can deduce the regularity theorems with respect tothe partial differential operators of homogeneous type and Isakov[2] obtained the same results with respect to partial differential operators of semihomogeneous type.

In this paper we are concerned with the partial differential operator:

$$P(x,D_x)=\sum_{j=1}^{q-1}a_j(x)D_j+\sum_{j=q}^{n}a_j(x)D_j^2+\sum_{j=q}^{n}b_j(x)D_j+c(x), \tag{1.1}$$

where a_j, b_j, and c are contained in $L_\infty(\bar{\Omega})$, and each $a_j \in C^1(\bar{\Omega})$. We assume that all coefficients are real valued functions defined in a bounded open set $\Omega \subset R^n$.

Now we introduce some notations and pseudo-convex surface with respect to semihomogeneous partial differential operators. For some multi-indices $m=(m_1,\cdots,m_n)$ of positive integers with

$$0<m_1\le m_2\le\cdots\le m_{q-1}<m_q=\cdots=m_n,$$

and $\alpha=(\alpha_1,\cdots,\alpha_n)$ we write $|\alpha:m|=\sum_{j=1}^{n}\alpha_j/m_j$ and $D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}$ where $D_j=-i\ \partial/\partial x_j=-i\ \partial_j$. Let

$$R(x,D)=\sum_{|\alpha:m|\le 1}a_\alpha(x)D^\alpha$$

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We then denote the principal part of $R(x, D)$ and its principal symbol by

$$\sum_{|\alpha:m|=1} a_\alpha(x) D^\alpha \quad \text{and} \quad \sum_{|\alpha:m|=1} a_\alpha(x)\xi^\alpha,$$

respectively.

DEFINITION 1.1. *We say that a partial differential operator* $P(x, D) = \sum_{|\alpha:m|\leq 1} a_\alpha(x) D^\alpha$ *is (d-)semihomogeneous of degree* M *if the principal symbol* $p_M(x,\xi) \equiv \sum_{|\alpha:m|=1} a_\alpha(x)\xi^\alpha$ *of* P *satisfies that*

$$p_M(x, t^{d_1}\xi_1, t^{d_2}\xi_2, \ldots, t^{d_n}\xi_n) = t^M p_M(x,\xi)$$

for $t > 0$ *and* $(x,\xi) \in R^n_x \times R^n_\xi$, *where* $d = (d_1, \cdots, d_n)$, $d_j = m_n/m_j$ *is an n-tuple of positive numbers.*

We note that the partial differential operator (1.1) is semihomogeneous of degree 2 with $d = (2, \ldots, 2, 1, \ldots, 1)$ and that $p_2(x,\xi) = \sum_{|\alpha:m|=1} a_\alpha(x)\xi^\alpha$ with $m_1 = \cdots = m_{q-1} = 1, m_q = \cdots = m_n = 2$. We shall denote by $[\xi]_d$ the function defined implicitly by the relation $\sum_{j=1}^n \frac{\xi_j^2}{[\xi]_d^{2d_j}} = 1$, if $\xi \neq 0$ and $[0]_d = 0$. The function $[\xi]_d$ is semihomogeneous of degree 1. Thus there exists a constant $C > 0$ satisfying

$$C^{-1} \sum_{j=1}^n |\xi_j|^{1/d_j} \leq [\xi]_d \leq C \sum_{j=1}^n |\xi|^{1/d_j}.$$

DEFINITION 1.2. *If* $f(x,\xi)$, $g(x,\xi)$ *be* C^1 *functions. The semihomogeneous Poisson bracket of* f, g *is defined by*

$$\{f, g\}_q = \sum_{j=q}^n \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}.$$

We assume that ϕ be a real valued function in $C^\infty(\bar{\Omega})$ such that

$$\nabla_q \phi \equiv (0, \cdots, 0, \partial_q \phi, \cdots, \partial_n \phi)$$

in Ω.

DEFINITION 1.3. *Let $\phi \in C^\infty(\Omega)$ with $\nabla_q \phi(x_0) \neq 0$. The hypersurface*

$$S = \{\phi(x) = \phi(x_0)\}$$

is called pseudo-convex with respect to P at x_0 if
(1) It is not characteristic of P,
(2) For all $\xi \in R^n \setminus 0$ $p_2(x_0, \xi) = \{p_2, \phi\}_q(x_0, \xi) = 0$ implies

$$\{p_2, \{p_2, \phi\}_q\}_q(x_0, \xi) > 0.$$

§2. RESULTS

2.1. Energy Inequality with Weight.
We estimate the L^2 norm with weight $\exp \tau\phi, \tau \in R$ with respect to P_2 by means of commutator [,].

LEMMA 2.1. *There exists a constant $C > 0$ such that for $u \in C_0^\infty$*

(2.1)
$$\int |P_2(x, D)u|^2 \exp 2\tau\phi \, dx \geq \tau \int S_\tau(x, , D) v\bar{v} \, dx - C(\int (1 + [\xi]_d^2) \, |\hat{v}(\xi)|^2 \, dx + \tau^2 \, ||v||^2)$$

where $[\xi]_d = \sum_{j=1}^{n} |\xi_j|^{m_j/m_n}$, $v = u \exp \tau\phi$ and $S_\tau(x, D)$ is the semihomogeneous partial differential operator of degree 2:

$$s_\tau(x, \xi) = \{p_2, \{p_2, \phi\}_q\}_q + \frac{\tau^2}{2}\{\{p_2, \phi\}_q, \{\{p_2, \phi\}_q, \phi\}_q\}_q. \tag{2.2}$$

Proof. Let $v = u \ \exp \tau\phi$. Then it follows that

$$\int |P_2(x,D)u|^2 \exp 2\tau\phi \, dx = \int |(\exp \tau\phi \, P_2(x,D) \ \exp -\tau\phi) \, v|^2 \, dx$$
$$\equiv \int |P_\tau(x,D)v|^2 \, dx.$$

It is obvious that $P_\tau(x,D) = P_2 + \tau[\phi, P_2] + \frac{\tau^2}{2}[\phi,[\phi,P_2]]$. Let $Q_1(x,D)$ be the partial differential operator with symbol $\{p_2,\phi\}_q$. We then obtain $[\phi, P_2] = i\,Q_1(x,D) - P_2(x,D)\phi$. Since

$$\{\{p_2,\phi\}_q,\phi\}_q = \sum_{j,k=q}^{n} \frac{\partial^2}{\partial\xi_j \partial\xi_k} p_2(x,\xi) \frac{\partial}{\partial x_j}\phi(x) \frac{\partial}{\partial x_k}\phi(x),$$

it follows that $[\phi,[\phi,P_2]] = -Q_0(x) \equiv -2p_2(x, \nabla_q \phi(x))$. So we have

$$P_\tau(x)(x,D) = P_2(x,D) + i\,\tau\,Q_1(x,D) - \frac{\tau^2}{2} Q_0(x) - \tau\,P_2(x,D)\phi.$$

On the other hand It is not difficult to show that

$$\int |P_\tau(x,D)\,v|^2\,dx \geq \frac{1}{2} \int |L_\tau(x,D)\,v|^2\,dx - c\,\tau^2\,||v||^2 \tag{2.3}$$

where $c = Max_x \ |P_2(x,D)\,\phi|$ and $L_\tau(x,D) = P_2(x,D) + i\,Q_1(x,D) - \frac{\tau^2}{2}Q_0(x)$. If $L_\tau^*(x,D)$ is the adjoint of $L_\tau(x,D)$ with respect to inner product $\int u \ \bar{v}\,dx$, it follows that

(2.4)

$$\int |L_\tau(x,D) \ v|^2\,dx = \int (|L_\tau(x,D) \ v|^2 - |L_\tau^*(x,D) \ v|^2)\,dx$$
$$= \int [L_\tau^*(x,D), \ L_\tau(x,D)] \ v\bar{v}\,dx.$$

Since coefficients of P_2 are real, we have

$$L_\tau^*(x,D) = P_2(x,D) - i\,\tau\,Q_1(x,D) - \frac{\tau^2}{2}Q_0(x) + R_1(x,D)$$
$$- i\tau R_0(x),$$

where

$$R_1(x,D) = P_2^*(x,D) - P_2(x,D),$$
$$R_0(x) = Q_1^*(x,D) - Q_1(x,D).$$

We note that $r_1(x,\xi)$ contains no variables $\xi_j,\ j=1,\dots,q-1.$ and that the principal symbol of

$$i\,[P_2(x,D),\ Q_1(x,D)] + \frac{\tau^2}{2}\ i\,[Q_1(x,,D),\ Q_0(x)]$$

equals

$$s_\tau(x,\xi) = \{p_2,\{p_2,\ \phi\}_q\}_q + \frac{\tau^2}{2}\ \{\{p_2,\ \phi\}_q,\{\{p_2,\ \phi\}_q,\ \phi\}_q\}_q.$$

Thus there exists $R_1'(x,D)$ of order ≤ 1 such that

$$i\,[P_2(x,D),\ Q_1(x,D)] + \frac{\tau^2}{2}\ i\,[Q_1(x,,D),\ Q_0(x)] = S_\tau(x,D) + R_1'(x,D).$$

It is easy to show that there exist $R_2'',\ R_1'',\ R_0''$ semihomogeneous of degree $\leq 2,\ \leq 1,\ 0$, respectively such that

$$[R_1(x,D) - i\ \tau\ R_0(x),\ L_\tau(x,D)] = R_2''(x,D) + \tau\ R_1''(x,D) + \tau^2\ R_0''(x).$$

Thus the Poisson bracket $[L_\tau^*(x,D),\ L_\tau(x,D)]$ equals

(2.5) $$2\tau\ S_\tau(x,D) + R_2''(x,D) + \tau\ R_1'''(x,D) + \tau^2\ R_0''(x),$$

where $R_1'''(x,D) = 2\ R_1'(x,D) + R_1''(x,D)$. Consequently from (2.3), (2.4), (2.5) we have

(2.6)

$$\int |P_2(x,D)\ u|^2\ \exp\ 2\tau\phi\ dx \geq \tau \int S_\tau\ (x,D)\ v\ \bar{v}\ dx + \frac{1}{2}\int R_2''(x,D)\ v\ \bar{v}\ dx + \frac{\tau}{2}\int R_1'''(x,D)\ v\ \bar{v}\ dx + \tau^2 \int R_0''(x)\ v\ \bar{v}\ dx - c\ \tau^2\ ||v||^2.$$

The inequality (2.1) follows from Cauchy-Schwarz inequality and semi-homogeneity of $R_2'',\ R_1''',\ R_0''$.

LEMMA 2.2. *Let the real valued function $\phi \in C^{\infty}(\bar{\Omega})$ satisfy $\nabla_q \phi \neq 0$ in $\bar{\Omega}$. Then it follows that*

(2.7)

$$p_2(x,\ \zeta) = p_2(x,\xi) - \frac{\tau^2}{2}\{\{p_2,\ \phi\}_q, \phi\}_q + i\ \tau\ \{p_2,\ \phi\}_q,$$

(2.8)

$$s_\tau(x,\ \xi) = \sum_{j,k=q}^{n} \frac{\partial^2\phi(x)}{\partial x_j \partial x_k}\frac{\partial}{\partial \xi_j}\ p_2(x,\zeta)\overline{\frac{\partial}{\partial \xi_k}\ p_2(x,\zeta)}$$
$$\frac{1}{\tau}\ Im \sum_{j=q}^{n} \frac{\partial}{\partial x_j}\ p_2(x,\ \zeta)\overline{\frac{\partial}{\partial \xi_j}\ p_2(x,\ \zeta)},$$

where $\zeta = \xi + i\ \tau\ \nabla_q\ \phi(x)$.

REMARK 2.3. *It is obvious that*

$$s_\tau\ (x,\ \xi) = \frac{1}{\tau}\ \{Re\ p_2\ (x,\ \zeta),\ Im\ p_2\ (x,\ \zeta)\}_q$$
$$= \frac{1}{2i\tau}\ \{\overline{p_2\ (x,\ \zeta)},\ p_2\ (x,\ \zeta)\}_q$$

2.2. Carleman Estimates.
We note that $p_2(x,\ \zeta)$ is the symbol of $L_\tau\ (x, D)$ and $p_2(x,\ \zeta) = 0$ if and only if $p_2\ -\ \frac{\tau^2}{2}\ \{\{p_2,\ \phi\}_q,\ \phi\}_q = \{p_2,\ \phi\}_q = 0$. From Lemma (2.1) and Lemma (2.2) we obtain:

THEOREM 2.4. *We assume that*

(2.9) $$p_2\ (x,\ \xi + i\tau\ \nabla_q\ \phi(x)) = 0 \Longrightarrow s_\tau\ (x,\ \xi) > 0.$$

Then there exist a constant $K > 0$ and a real number $\tau_0 > 0$ such that

(2.10) $$\tau^3 \int |u|^2\ \exp\ 2\tau\phi\ dx + \tau \sum_{j=q}^{n} \int |D_j u|^2\ \exp\ 2\tau\phi\ dx$$
$$\leq K \int |P_2(x,D)\ u|^2\ \exp\ 2\tau\phi\ dx$$

for all $u \in C_0^\infty(\Omega)$ *and for* $\tau \geq \tau_0$.

Proof. We assume that $\partial_n \phi(x_0) \neq 0$. By invariance of the condition (2.9) under the diffeomorphism $x = x(y)$ with $x_j - (x_0)_j = y_j, \quad j = 1, \dots, n-1$ we may assume that $\phi(x) - \phi(x_0) = x_n$ and $\phi(x_0) = 0$. From (2.9) it follows that $s_\tau(0,\ \xi) > 0$ in a set $V = \{(\xi,\ \tau) | [\zeta]_d = 1 \quad p_2(x, \zeta) = 0\}$ where $N = (0, \cdots, 0, 1)$ and $\zeta = \xi + i\tau N$. By continuity and semihomogeneity there exists a constant $C > 0$ such that

$$C\ [\zeta]_d^2 \leq s_\tau(0,\ \xi) + \frac{p_2(0, \zeta)}{[\zeta]_d^2}. \tag{2.11}$$

Thus for $v = u \ \exp\ \tau\phi \in C_0^\infty(\Omega)$ we multiply (2.11) by $|\hat{v}(\xi)|^2$ and integrate, which gives

$$(2\pi)^{-n} \int [\zeta]_d^2\ |\hat{v}(\xi)|^2\ d\xi \leq C_1 \{ \int S_\tau(0, D)\ v\ \bar{v}\ dx + (2\pi)^{-n} \int [\zeta]_d^{-2}\ |p_2(0, \zeta)|^2\ |\hat{v}(\xi)|^2\ d\xi \}. \tag{2.12}$$

It is not difficult to show that

$$\tau^2 \int |u|^2\ \exp\ 2\tau\phi\ dx + \sum_{j=q}^{n} \int |D_j u|^2\ \exp\ 2\tau\phi\ dx \leq (2\pi)^{-n} \int [\zeta]_d^2\ |\hat{v}(\xi)|^2\ d\xi. \tag{2.13}$$

Thus our theorem follows from (2.1), (2.13) for sufficiently large τ.

COROLLARY 2.5. *There exist positive constants* K *and* τ_0 *such that*

$$\tau^3 \int |u|^2\ \exp\ 2\tau\phi\ dx + \tau \sum_{j=q}^{n} \int |D_j u|^2\ \exp\ 2\tau\phi\ dx \leq K \int |P(x, D)u|^2\ \exp\ 2\tau\phi\ dx,$$

for all $u \in C_0^\infty$ *and* $\tau \geq \tau_0$.

THEOREM 2.6. *We assume that ϕ is pseudo-convex with respect to P at x_0. Then it follows that*

$$\tau^3 \int |u|^2 \exp 2\tau\phi \, dx + \tau \sum_{j=q}^{n} \int |D_j u|^2 \exp 2\tau\phi \, dx$$
$$\leq K \{\int |P(x,D)u|^2 \exp 2\tau\phi \, dx + \tau^3 \int |u|^2 \exp 2\tau\phi \, dx\}.$$

Proof. From the fact that there exist positive constants C_1, C_2 and C_3 such that

$$[\zeta]_d^2 \leq C_1 \, s_\tau(0, \, \xi) + C_2 \, \frac{p_2(0,\zeta)}{[\zeta]_d^2}. + C_3 \, \tau^2$$

our theorem follows.

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