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NONEXISTENCE OF POSITIVE SOLUTIONS

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1. Introduction.

The existence of positive solutions in a bounded domain of the following nonlinear elliptic equation has been extensively studied:

$$\begin{aligned} -\Delta u = u^p + f(x,u) \quad & \text{on} \quad \Omega, \\ u > 0 \quad & \text{on} \quad \Omega \\ u = 0 \quad & \text{on} \quad \partial\Omega \end{aligned}$$

where $p = (n+2)/(n-2)$, $f(x,0) = 0$ and $f(x,u)$ is a lower-order perturbation of u^p in the sense that $\lim_{u\to\infty} f(x,u)/u^p = 0$. See [BL], [AS],[L].

In this paper, we are concerned with the nonexistence of positive solutions of the equation

$$-\Delta u = \lambda u + u|u|^{2^*-2} \quad \text{in } \Omega, \tag{1.1}$$

$$u > 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } \Omega, \tag{1.3}$$

where $2^* = 2n/(n-2)$.

2. Main Result

Lemma 2.1. *Let V be a vector field on $\mathbb{R}^n$. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous with primitive $G(u) = \int_0^u g(v)dv$ and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the following equation*

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$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

in a domain $\Omega \subset\subset \mathbb{R}^n$. Then there holds

$$\frac{1}{2}\int_{\partial\Omega} |\frac{\partial u}{\partial \nu}|^2 \langle \nu, V\rangle d\sigma + \int_{\Omega} (\frac{|\nabla u|^2}{2} - G(u)\ ) \operatorname{div} V dx$$
$$- \sum_{j=1}^{n} \int_{\Omega} \langle \nabla u, \nabla V_j\rangle \frac{\partial u}{\partial x_j} dx = 0,$$

where ν denotes the exterior unit normal.

Proof.

$$\begin{aligned} \operatorname{div}(\nabla u\ \langle V, \nabla u\rangle\ ) &= \Delta u\ \langle V, \nabla u\rangle + \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial V_j}{\partial x_i}\frac{\partial u}{\partial x_j} + \langle V, \nabla(\frac{|\nabla u|^2}{2})\rangle \\ &= \nabla u\ \langle V, \nabla u\rangle + \sum_{j+1}^{n} \langle \nabla u, \nabla V_j\rangle \frac{\partial u}{\partial x_j} + \langle V, \nabla(\frac{|\nabla u|^2}{2})\rangle. \end{aligned}$$

Hence

$$\nabla u\ \langle V, \nabla u\rangle = \operatorname{div}(\nabla u\ \langle V, \nabla u\rangle) - \sum_{j=1}^{n} \langle \nabla u, \nabla V_j\rangle \frac{\partial u}{\partial x_j} - \langle v, \nabla(\frac{|\nabla u|^2}{2})\rangle.$$

Therefore

$$\begin{aligned} 0 &= (\Delta u + g(u))\langle V, \nabla u\rangle \\ &= \Delta u\ \langle V, \nabla u\rangle + g(u)\langle V, \nabla u\rangle \\ &= \operatorname{div}(\nabla u\ \langle V, \nabla u\rangle) - \sum_{j=1}^{n} \langle \nabla u, \nabla V_j\rangle \frac{\partial u}{\partial x_j} - \langle V, \nabla(\frac{|\nabla u|^2}{2})\rangle \\ &\quad + \langle V, \nabla G(u)\rangle \\ &= \operatorname{div}(\nabla u\ \langle V, \nabla u\rangle - \frac{|\nabla u|^2}{2} V + G(u)V\ ) \\ &\quad + \frac{|\nabla u|^2}{2} \operatorname{div} V - G(u) \operatorname{div} V - \sum_{j=1}^{n} \langle \nabla u, \nabla V_j\rangle \frac{\partial u}{\partial x_j}. \end{aligned}$$

Observe that

$$\langle x, \nabla u\rangle = \langle x, \nu\rangle \frac{\partial u}{\partial \nu} \quad \text{and} \quad |\nabla u|^2 = |\frac{\partial u}{\partial \nu}|^2 \quad \text{on} \quad \partial\Omega.$$

Integrating this identity over Ω we have

$$\frac{1}{2}\int_{\partial\Omega} |\frac{\partial u}{\partial \nu}|^2 \langle \nu, V\rangle d\sigma + \int_{\Omega} (\frac{|\nabla u|^2}{2} - G(u)\ ) \operatorname{div} V dx - \sum_{j=1}^{n} \int_{\Omega} \langle \nabla u, \nabla V_j\rangle \frac{\partial u}{\partial x_j} dx = 0.$$

We now apply this Lemma to obtain the following nonexistence result.

THEOREM 2.1. *Suppose $0 \in \Omega \neq \mathbb{R}^n$ is a smooth (possibly unbounded) domain in $\mathbb{R}^n, n \geq 3$. Suppose Ω has a vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ such that*

(1) $\langle \nu, V\rangle > 0$ *on* $\partial\Omega$,
(2) $\operatorname{div} V = c$, *c is a positive constant,*
(3) *There is a positive constant M such that*

$$\sum_{i,j}^{n} \frac{\partial V_j}{\partial x_j} \xi_i \xi_j \leq M|\xi|^2,$$

(4) $nM \leq c$.

Then the boundary value problem (1.1), (1.2) and (1.3) has no positive solutions in H_0^1.

Proof. Let $g(u) = \lambda u + u|u|^{2^*-2}$ with

$$G(u) = \frac{\lambda}{2}|u|^2 + \frac{1}{2^*}|u|^*.$$

We know that any solution of (1.1) and (1.3) is smooth on $\overline{\Omega}$. Hence by Lemma 2.1

(2.1)

$$\frac{1}{2}\int_{\partial\Omega} |\frac{\partial u}{\partial \nu}|^2 \langle \nu, V\rangle d\sigma + \frac{c}{2}\int_{\Omega} (|\nabla u|^2 - \lambda |u|^2 - \frac{2}{2^*}|u|^{2^*})dx$$

$$= \int_{\Omega} (\sum_{i,j=1}^{n} \frac{\partial V_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j})dx.$$

Note that if u is a solution of (1.1), then

$$\int_{\Omega} (|\nabla u|^2 - \lambda|u|^2 - |u|^{2^*})dx = 0. \tag{2.2}$$

Multiplying both sides of (2.1) by n/c and using the fact (2.2), we obtain

$$\frac{n}{2c}\int_{\partial\Omega} |\frac{\partial u}{\partial \nu}\langle \nu, V\rangle + \frac{n-2}{2}\int_{\Omega} |\nabla u|^2 dx + \frac{n}{2}|\lambda| \int_{\Omega} |u|^2 dx$$

$$- \frac{n-2}{2}\int_{\Omega} |u|^{2^*} dx = \frac{n}{c}\int \Omega(\sum_{i,j=1}^{n} \frac{\partial V_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j})dx - \int_{\Omega} |\nabla u|^2 dx.$$

Hence

$$\begin{aligned}
&\frac{n}{2c}\int_{\partial\Omega} |\frac{\partial u}{\partial \nu}\langle \nu, V\rangle d\sigma + |\lambda| \int_{\Omega} |u|^2 dx \\
&= \frac{n}{c}\int_{\Omega} (\sum_{i,j=1}^{n} \frac{\partial V_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j})dx - \int_{\Omega} |\nabla u|^2 dx \\
&\leq (\frac{nM}{c} - 1)\int_{\Omega} |\nabla u|^2 dx \\
&\leq 0.
\end{aligned}$$

Since $\langle \nu, V\rangle > 0$ on $\partial\Omega$, we must have

$$\frac{n}{2c}\int_{\partial\Omega}|\frac{\partial u}{\partial\nu}|^2\langle\nu,V\rangle d\sigma+|\lambda|\int_\Omega|u|^2dx=0.$$

Thus $\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega$, and hence $u\equiv 0$ by the principle of unique continuation. This completes the proof.

References

[AS] A. Ambroseti and M.Struwe, *A note on the problem* $-\Delta=\lambda u+u|u|^{2^*-2}$, Manuscripta Math. **54** (1986), 373–379.

[BN] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical sobolev exponents*, Comm.Pure Appl. Math. **36** (1983), 437–476.

[L] Z.Liqun, *Uniqueness of positive solutions of* $\Delta u+u+u^p=0$ *in a ball*, Comm. Partial Diff. Eq. **17(7&8)** (1992), 1141–1164.

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