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SOME RESULTS OF OPERATORS IN THE CLASS $\mathbb{A}^l_{m,n}$

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I. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert spaces and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\perp_{\mathcal{A}_T}$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\perp_{\mathcal{A}_T}$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by Q_T. One knows that $\mathcal{A}_T$ is the dual space of Q_T and that the duality is given by

$$(1) \qquad \langle A, [L]\rangle = tr(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T.$$

The Banach space Q_T is called a predual of $\mathcal{A}_T$. For x and y in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(2) \qquad (x \otimes y)(u) = (u, y)x \quad for \quad all \quad u \in \mathcal{H}.$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $\mathbb{A}_{m,n}$ (to be defined in section 2) were defined by Bercovici-Foias-Pearcy in [2]. Also these classes are closely related to the study of the theory of dual algebras. In 1991, Chevreau-Pearcy [7] studied for the first time common noncyclic

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vectors for the families of operators in order to solve the invariant subspace problem of bounded operators whose spectral radius is one. As a sequel to this study, in this paper we consider some dilation theories for the membership in these classes $\mathbb{A}^l_{m,n}$ (to be defined in section 3).

2. Notation and Preliminaries

Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0), T will be called an ***absolutely continuous contraction.*** We denote by $\mathbb{A} = \mathbb{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Foias-Sz.Nagy functional calculus $\Phi_T : \mathcal{H}^\infty \to \mathcal{A}_T$ is an isometry. ([3], Theorem 4.1)

DEFINITION 2.1. ([10]) Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let m and n be any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, \quad 0 \leq j < n, \tag{3}$$

where $\{[L_{ij}]\}_{\substack{0\leq i<m \\ 0\leq j<n}}$ is an arbitrary $m \times n$ array from $Q_{\mathcal{A}}$, has a solution $\{x_i\}_{0\leq i<m}, \{y_j\}_{0\leq j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if m and n are positive integers and r is a fixed real number satisfying $r \geq 1$, a dual algebra $\mathcal{A}$ (with property $(\mathbb{A}_{m,n})$) is said to have property $(\mathbb{A}_{m,n}(r))$ if for every $s > r$ and every $m \times n$ array $\{[L_{i,j}]\}_{\substack{0\leq i<m \\ 0\leq j<n}}$ from $Q_{\mathcal{A}}$, there exist sequences $\{x_i\}_{0\leq i<m}$ and $\{y_j\}_{0\leq j<n}$ from $\mathcal{H}$ that satisfy (3) and also satisfy the following conditions:

$$\|x_i\|^2 \leq s \sum_{0\leq j<n} \|[L_{ij}]\|, \quad 0 \leq i < m, \tag{4a}$$

and

$$\|y_j\|^2 \leq s \sum_{0\leq i<m} \|[L_{ij}]\|, \quad 0 \leq j < n. \tag{4b}$$

For brief notation, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. Furthermore, if m and n are cardinal numbers such that $1 \le m, n \le \aleph_0$, we denote by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$.

DEFINITION 2.2. ([13]) Let $\mathbb{N}$ be the set of all natural numbers. If $n \in \mathbb{N}$, we denoted by $\tilde{\mathcal{H}}^{(n)}$ the Hilbert space consisting of the direct sum of n copies of $\mathcal{H}$ and by $T^{(n)}$ the n-fold *ampliation* of T acting on $\tilde{\mathcal{H}}^{(n)}$ defined by

$$T^{(n)}(x_1 \oplus \cdots \oplus x_n) = Tx_1 \oplus \cdots \oplus Tx_n. \tag{5}$$

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra. We define

$$\mathcal{A}^{(n)} = \overbrace{A \oplus \cdots \oplus A}^{n \text{ copies}}, A \in \mathcal{A}.$$

Then $\mathcal{A}^{(n)}$ is indeed a dual algebra on $\tilde{\mathcal{H}}^{(n)}$. For each T in $\mathcal{L}(\mathcal{H})$ it is clear that $(\mathcal{A}_T)^{(n)} = \mathcal{A}_{T^{(n)}}$. We shall employ the notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contractions T in $\mathcal{L}(\mathcal{H})$ such that the sequences $\{T^{*n}\}$ converges to zero in the strong operator topology and is denoted by, as usual, $C_{0\cdot} = (C_{\cdot 0})^*$.

3. MAIN RESULTS

DEFINITION 3.1. ([9]) Let m, n and l be any cardinal numbers such that $1 \le m, n, l \le \aleph_0$. We denote by $\mathbb{A}^{l}_{m,n}(\mathcal{H})$ the class of all $\{T_k\}_{k=1}^{l}$ in $\mathbb{A}(\mathcal{H})$ such that every $m \times n \times l$ system of simultaneous equations of the form

$$[x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k}, \tag{6}$$

where $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0\le i<m \\ 0\le j<n}}$ is an arbitrary $m \times n$ array from Q_{T_k} for each $k = 1, \cdots, l$, has a solution $\{x_i\}_{0\le i<m}, \{y_j^{(k)}\}_{\substack{0\le j<n \\ 1\le k\le l}}$ consisting of a pair

of sequences of vectors from $\mathcal{H}$. Furthermore, if for every doubly indexed family $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0\le i<m\\0\le j<n}}$ of elements of Q_{T_k} for each $k=1,\cdots,l$, such that the rows and columns of the matrix $(\|[L_{ij}^{(k)}]_{T_k}\|)$ are summable and r is a fixed real number satisfying $r\ge 1$, then we denoted by $\mathbb{A}_{m,n}^{l}(r)$ the class of all $\{T_k\}_{k=1}^{l}$ in $\mathbb{A}(\mathcal{H})$ such that every $m\times n\times l$ system of simultaneous equations of the form (6) has a solution $\{x_i\}_{0\le i<m}, \{y_j^{(k)}\}_{\substack{0\le j<n\\1\le k\le l}}$ consisting of a pair of sequences of vectors from $\mathcal{H}$ and also satisfy the following conditions, for every $s>r$,

$$(7) \qquad \|x_i\|^2 \le s \sum_{0\le j<n} \|[L_{ij}^{(k)}]_{T_k}\|, \quad 0\le i<m, 1\le k\le l$$

and

$$(8) \qquad \|y_j^{(k)}\|^2 \le s \sum_{0\le i<m} \|[L_{ij}^{(k)}]_{T_k}\|, \quad 0\le j<n, 1\le k\le l.$$

LEMMA 3.2. *([12]) Suppose $T,S\in\mathbb{A}(\mathcal{H})$. Let $[L]_T\in Q_T$ and let $[M]_S\in Q_S$. Then*

$$\phi_T([L]_T)=\phi_S([M]_S)$$

if and only if

$$\langle T^n,[L]_T\rangle=\langle S^n,[M]_S\rangle, n=0,1,2,\cdots.$$

We are ready to prove main results. Suppose $\mathcal{H}_r$ is a separable infinite dimensional complex Hilbert space for $r=1,2,\cdots,m$.

THEOREM 3.3. *Assume $\{T_k^{(r)}\}_{k=1}^m\in\mathbb{A}_{1,1}^m(\mathcal{H}_r)$ for each $r=1,2,\cdots,m$. Let $\mathcal{H}=\oplus_{r=1}^m\mathcal{H}_r$ and $T_k=\oplus_{r=1}^m T_k^{(r)}$. Then $\{T_k\}_{k=1}^m\in\mathbb{A}_{1,m}^m(\mathcal{H})$.*

Proof. We first show that $T_k\in\mathbb{A}(\mathcal{H})$ for all k. If $f\in\mathbb{H}^\infty(\mathbb{D})$, then $\|f(T_k)\|=\sup_r\|f(T_k^{(r)})\|=\|f\|_\infty$, since $T_k^{(r)}\in\mathbb{A}(\mathcal{H}_r)$ for $k=1,2,$

$\cdots, m$. To show that $\{T_k\}_{k=1}^m \in \mathbb{A}^m_{1,m}(\mathcal{H})$, suppose we are given array $\{[L_r^{(k)}]_{T_k}\}_{1\le r\le m}$ in Q_{T_k} for each $k = 1, \cdots, m$. Let us consider the usual isometric isomorphic weak* homeomorphism $\phi_{k,r}$ from Q_{T_k} onto $Q_{T_k^{(r)}}$. Since $\phi_{k,r}([L_r^{(k)}]_{T_k}) \in Q_{T_k^{(r)}}$, by definition, for each r, there exist vectors $x^{(r)}$ and $\{y_k^{(r)}\}_{k=1}^m$ in $\mathcal{H}_r$ such that

$$\phi_{k,r}([L_r^{(k)}]_{T_k}) = [x^{(r)} \otimes y_k^{(r)}]_{T_k^{(r)}}, \quad 1 \le k \le m. \tag{9}$$

Letting $\tilde{x} = \oplus_{r=1}^m x^{(r)}$ and let $\tilde{w}_r^{(k)}$ be the vector in $\mathcal{H}$ that has $y_k^{(r)}$ as r-component and zeros elsewhere, for $1 \le r, k \le m$. Then $\tilde{x}, \tilde{w}_r^{(k)} \in \mathcal{H}$ and we have, by lemma 3.2 and (9),

$$\begin{aligned}\phi_{k,r}([L_r^{(k)}]_{T_k}) &= [x^{(r)} \otimes y_k^{(r)}]_{T_k^{(r)}}, \quad 1 \le k, r \le m \\ &= \phi_{k,r}([\tilde{x} \otimes \tilde{w}_r^{(k)}]_{T_k}).\end{aligned} \tag{10}$$

So we get

$$[L_r^{(k)}]_{T_k} = [\tilde{x} \otimes \tilde{w}_r^{(k)}]_{T_k} \tag{11}$$

for $1 \le r \le m$ and $1 \le k \le m$. Therefore, we have $\{T_k\}_{k=1}^m \in \mathbb{A}^m_{1,m}(\mathcal{H})$.

THEOREM 3.4. *Assume $\{T_k^{(r)}\}_{k=1}^m \in \mathbb{A}^m_{1,n}(\mathcal{H}_r)$ for each $r = 1, 2, \cdots, m$. Let $\mathcal{H} = \oplus_{r=1}^m \mathcal{H}_r$ and $T_k = \oplus_{r=1}^m T_k{}^{(r)}$. Then $\{T_k\}_{k=1}^m \in \mathbb{A}^m_{m,n}(\mathcal{H})$.*

Proof. The proof is similar to that of Theorem 3.3.

THEOREM 3.5. *Suppose $\mathcal{H}_w$ is a separable infinite dimensional complex Hilbert space for $w = 1, \cdots, m$. Suppose also that $\{T_k^{(w)}\}_{k=1}^m \in \mathbb{A}^m_{1,n_w}(\mathcal{H}_w)(r_w)$ where $n_w \in \mathbb{N}$ and $r_w \ge 1$ for all $w, 1 \le w \le m$. Let $\mathcal{H} = \oplus_{w=1}^m \mathcal{H}_w$ and $T_k = \oplus_{w=1}^m T_k^{(w)}$. Then, we have*

$$\{T_k\}_{k=1}^m \in \mathbb{A}^m_{m,N}(\mathcal{H})(r)$$

where $N = m(n_1 + \cdots + n_m)$ *and* $r = \max\{r_w : 1 \leq w \leq m\}$.

Proof. By theorem 3.3, $T_k \in \mathbb{A}(\mathcal{H})$ for all k. To show that $\{T_k\}_{k=1}^m \in \mathbb{A}_{m,N}^m(\mathcal{H})(r)$, let $s > r$ and suppose we are given array $\{[L_{wj}^{(k)}]_{T_k}\}_{\substack{1 \leq w \leq m \\ 1 \leq j \leq N}}$ in Q_{T_k} for each $k = 1, \cdots, m$. Let us denote $\phi_{k,w} = \phi_{T_k^{(w)}}^{-1} \circ \phi_{T_k}$. Then it is obvious that $\phi_{k,w}$ is an isometric isomorphic weak* homeomorphism from Q_{T_k} onto $Q_{T_k^{(w)}}$. Since $\phi_{k,w}([L_{wj}^{(k)}]_{T_k}) \in Q_{T_k^{(w)}}$, for any $s > r_w$, there exist a vector $x^{(w)}$ and sequence of vectors $\{y_j^{(k,w)}\}_{j=1,k=1}^{n_w,m}$ in $\mathcal{H}_w, 1 \leq w \leq m$ such that

$$\phi_{k,w}([L_{wj}^{(k)}]_{T_k}) = [x^{(w)} \otimes y_j^{(k,w)}]_{T_k^{(w)}} \tag{12}$$

for j and $k, 1 \leq j \leq n_w, 1 \leq w \leq m$ and furthermore,

$$\|x^{(w)}\|^2 \leq s \sum_{j=1}^{n_w} \|[L_{wj}^{(k)}]_{T_k}\|, \quad 1 \leq k \leq m, \tag{13}$$

and

$$\|y_j^{(k,w)}\|^2 \leq s \sum_{w=1}^{m} \|[L_{wj}^{(k)}]_{T_k}\|, \quad 1 \leq k \leq m. \tag{14}$$

Now let $\tilde{x}_w$ be the vector in $\mathcal{H}$ that has $x^{(w)}$ as w-component and zeros elsewhere and let $\tilde{v}_j^{(k)}$ be the vector in $\mathcal{H}$ that has $y_j^{(k,w)}$ as w-component and zeros elsewhere, for $1 \leq j \leq n_w, 1 \leq w, k \leq m$. Then it follows from Lemma 3.2 and (12) that we have

$$\begin{aligned} &\phi_{k,w}([L_{wj}^{(k)}]_{T_k}) \\ &= [x^{(w)} \otimes y_j^{(k,w)}]_{T_k^{(w)}} \\ &= \phi_{k,w}([\tilde{x}_w \otimes \tilde{v}_j^{(k)}]_{T_k}), \end{aligned} \tag{15}$$

and for $1 \leq j \leq n_w, 1 \leq w, k \leq m$. So we get

$$[L_{wj}^{(k)}]_{T_k} = [\tilde{x}_w \otimes \tilde{v}_j^{(k)}]_{T_k} \tag{16}$$

for $1 \leq w, k \leq m, 1 \leq j \leq n_1 + \cdots + n_m$. Furthermore, we have

$$\|\tilde{x}_w\|^2 = \|x^{(w)}\|^2 \leq s \sum_{j=1}^{n_w} \|[L_{wj}^{(k)}]_{T_k}\| \leq s \sum_{j=1}^{N} \|[L_{wj}^{(k)}]_{T_k}\|, \quad 1 \leq k \leq m, \tag{17}$$

and

$$\|\tilde{v}_j^{(k)}\|^2 = \|y_j^{(k,w)}\|^2 \leq s \sum_{w=1}^{m} \|[L_{wj}^{(k)}]_{T_k}\|, \quad 1 \leq k \leq m. \tag{18}$$

Therefore, we have

$$\{T_k\}_{k=1}^m \in \mathbb{A}_{m,N}^m(\mathcal{H})(r),$$

where $r = \max\{r_w : 1 \leq w \leq m\}$.

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