Kangweon-Kyungki Math. Jour. 3 (1995), No. 1, pp. 59-70

THE NONLINEAR SUSPENSION BRIDGE EQUATION WITH A ROTARY INERTIA TERM

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0. Introduction.

In this paper we investigate the existence of solutions of a nonlinear suspension bridge equation, in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R$, of the type

(0.1)
$$-K_1 u_{xxtt} + u_{tt} + K_2 u_{xxxx} + K_3 u^+ = 1 + k\cos x + \varepsilon h(x,t),$$
$$u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0.$$

The first term in (0.1), due to L. Rayleigh, represents the effect of rotary inertia, as can be traced from the derivation. In many applications, its effect is small.

McKenna and Walter [7] studied nonlinear oscillations in a nonlinear suspension bridge equation without the first term in (0.1)

(0.2)
$$u_{tt} + u_{xxxx} + bu^+ = 1 + \varepsilon h(x,t) \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R,$$
$$u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0.$$

This equation represented a bending beam supported by cables under a constant load $w = 1$. The constant b represented the restoring force if the cables were stretched. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression. They proved a counterintuitive result : if the cables were weak, that is, b is small then there was only a unique solution. However, if b was large (that

Received October 4, 1994.
* Supported in part by GARC-KOSEF and Inha Research Foundation.

is, the cables were strengthened), then large scale oscillatory periodic solutions existed.

In this paper we improve this result in two ways. First, we generalize the beam equation to include the effect of rotary inertia. Second, we use variational reduction method to study this suspension bridge equation.

In sections 1 and 2 we shall deal with the nonlinear bridge equation with constant coefficients, in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R$

(0.3)
$$-\frac{1}{4}u_{xxtt} + u_{tt} + u_{xxxx} + bu^{+} = 1 + k\cos x + \varepsilon h(x,t)$$
$$u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0,$$

where $4 < b < 19$. The effect of the first term in (0.1) is small, so we took the small coefficient $-\frac{1}{4}$ in (0.3). We shall assume that h in (0.3) is even in x and t and periodic with period π and we shall look for π-periodic solution of (0.3).

In section 3, we study equation (0.3) under a weak periodic condition, by a variational reduction method.

1. A Priori Bound.

Let L be the differential operator

$$Lu = -\frac{1}{4}u_{xxtt} + u_{tt} + u_{xxxx}.$$

The eigenvalue problem for $u(x,t)$

(1.1)
$$Lu = \lambda u \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R,$$
$$u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0,$$
$$u(x,t) = u(-x,t) = u(x,-t) = u(x, t+\pi)$$

has infinitely many eigenvalues λ_{mn} and corresponding eigenfunctions $\phi_{mn}(m, n \geq 0)$ given by

$$\lambda_{mn} = (2n+1)^4 - m^2((2n+1)^2 + 4), \quad (m, n = 0, 1, 2, \dots).$$
$$\phi_{mn} = \cos 2mt \cos(2n+1)x,$$

We remark that all eigenvalues in the interval $(-36, 29)$ are given by

$$\lambda_{29} = -19 < \lambda_{10} = -4 < \lambda_{00} = 1.$$

The normalized eigenfunctions are denoted by

$$\theta_{mn} = \frac{\phi_{mn}}{\|\phi_{mn}\|},$$

where $\|\phi_{mn}\| = \frac{\pi}{2}$ for $(m > 0)$, $\|\phi_{0n}\| = \frac{\pi}{\sqrt{2}}$. Let $\mathbb{Q}$ be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and H be the Hilbert space defined by

$$H = \{u \in L_2(\mathbb{Q}) : u \text{ is even on } x \text{ and } t\}.$$

Then the set of $\{\theta_{mn}\}$ is an orthonormal base in H.

We consider weak solutions of problems of the type

$$(1.2) \qquad \begin{gathered} Lu = f(u, x, t) \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \\ u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \end{gathered}$$

where u is even and π-periodic in t and even in x. A weak solution of (1.2), which is also called a solution in H, is of the form

$$u = \sum c_{mn}\theta_{mn} \quad \text{with} \quad Lu \in H;$$

i.e., $\sum \lambda_{mn}^2 c_{mn}^2$ is finite. Our function will be such that $u \in H$ implies $f(u, x, t) \in H$. The following symmetry theorem was proved in [10].

THEOREM 1.1. *Let $H = L_2(\Omega)$. Assume that $L : D(L) \to H$ is a linear, selfadjoint operator which possesses two closed invariant subspaces H_1 and $H_2 = H_1^{\perp}$. Let σ denote the spectrum of L and σ_i the spectrum of $L_{H_i} (i = 1, 2; \sigma = \sigma_1 \cup \sigma_2)$. Let $\frac{\partial f}{\partial u}(u, x) = f_u$ be piecewise smooth and assume that $f_u \in [a, b]$ for $u \in R$ and $x \in \Omega$.*

If $[a,b]\cap\sigma_2=\phi$ *and if the Nemytzki operator* $u\longmapsto Fu=f(u(s),x)$ *maps* H_1 *into itself, then every solution of*

$$Lu=f(u,x)\quad in\quad H$$

is in H_1.

LEMMA 1.1. *For* $-1<b<19$ *the problem*

$$Lu+bu^{+}=0\quad in\quad H \tag{1.3}$$

has only the trivial solution $u=0$.

We establish a *priori* bound for solutions of (0.3), namely,

$$Lu+bu^{+}=1+k\cos x+\varepsilon h,\qquad (k\geq 0)\text{ in }H. \tag{1.4}$$

LEMMA 1.2. *Let* $k\geq 0$ *be fixed. Let* $h\in H$ *with* $\|h\|=1$ *and* $\alpha>0$ *be given. Then there exists* $R_0>0$ *(depending only on* h *and* α*) such that for all* b *with* $-1+\alpha\leq b\leq 19-\alpha$ *and all* $\varepsilon\in[-1,1]$ *the solutions of (1.4) satisfy* $\|u\|<R_0$.

Proof. We shall apply Lemma 1.1. Assume Lemma 1.2 does not hold. Then there is a sequence (b_n,ε_n,u_n) with $b_n\in[\alpha-1,19-\alpha]$, $|\varepsilon_n|\leq 1$, $\|u_n\|\to\infty$ such that

$$u_n=L^{-1}(1+k\cos x+\varepsilon_n h-b_n u_n^{+}).$$

Put $w_n=\dfrac{u_n}{\|u_n\|}$. Then

$$w_n=L^{-1}\left(\frac{1}{\|u_n\|}+\frac{k}{\|u_n\|}\cos x+\frac{\varepsilon_n}{\|u_n\|}h-b_n w_n^{+}\right).$$

The operator L^{-1} is compact. Therefore we may assume that $w_n\to w_0$ and $b_n\to b_0\in(-1,19)$. Since $\|w_n\|=1$ for all n, $\|w_0\|=1$ and w_0 satisfies

$$w_0=L^{-1}(-bw_0^{+})\quad\text{or}\quad Lw_0+bw_0^{+}=0\text{ in }H.$$

This contradicts to Lemma 1.1 and proves Lemma 1.2. □

2. Existence of Solutions of a Nonlinear Suspension Bridge Equation with a Constant Coefficient.

The main result in this section is the following.

THEOREM 2.1. *Let $h \in H$ with $\|h\| = 1$ and $4 < b < 19$. Then there is $\varepsilon_0 > 0$ such that if $|\varepsilon| < \varepsilon_0$ the equation (1.4) has at least two solutions.*

In other words, the equation (0.3) has at least two π-periodic solutions. The proof of Theorem 2.1 requires several lemmas. First we discuss the Leray-Schauder degree d_{LS}.

LEMMA 2.1. *Under the assumptions and with notations of Lemma 1.2,*

$$d_{LS}(u - L^{-1}(1 + k\cos x - bu^+ + \varepsilon h), B_R, 0) = 1$$

for all $R \geq R_0$.

Proof. Let $b = 0$. Then we have

$$d_{LS}(u - L^{-1}(1 + k\cos x + \varepsilon h), B_R, 0) = 1$$

since the map is simply a translation of the identity and since $\|L^{-1}(1 + k\cos x + \varepsilon h)\| < R_0$ by Lemma 1.2. □

In case $b \neq 0(-1 < b < 19)$, the result follows in the usual way by invariance under homotopy, since all solutions are in the open ball B_{R_0}.

The following lemma was proved by McKenna and Walter [11].

LEMMA 2.2. *For $-1 < b$, the boundary value problem*

$$y^{(4)} + by = 1 \quad in \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y\left(\pm\frac{\pi}{2}\right) = y''\left(\pm\frac{\pi}{2}\right) = 0 \tag{2.1}$$

has a unique solution y, which is even in x, positive, and satisfies

$$y'\left(-\frac{\pi}{2}\right) > 0 \quad and \quad y'\left(\frac{\pi}{2}\right) < 0.$$ □

We can obtain an easy consequence of Lemma 2.2.

LEMMA 2.3. *Let $k \geq 0$ be fixed. For $-1 < b$ the boundary value problem*

(2.2)
$$y^{(4)} + by = 1 + k\cos x \quad in \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y\left(\pm\frac{\pi}{2}\right) = y''\left(\pm\frac{\pi}{2}\right) = 0$$

has a unique solution y, which is even and positive. Also the solution y satisfies

$$y'\left(-\frac{\pi}{2}\right) > 0 \quad and \quad y'\left(\frac{\pi}{2}\right) < 0.$$

Proof. The function

$$y_1 = y(x) - \frac{k}{1+b}\cos x$$

satisfies

(2.3)
$$y_1^{(4)} + by_1 = 1 \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$y_1\left(\pm\frac{\pi}{2}\right) = y_1''\left(\pm\frac{\pi}{2}\right) = 0.$$

By Lemma 2.2, we see that the solution y_1 is unique, even in x, positive, and satisfies

$$y_1'\left(-\frac{\pi}{2}\right) > 0 \quad \text{and} \quad y_1' < 0.$$

So the solution y is unique, even in x, positive and satisfies

$$y'\left(-\frac{\pi}{2}\right) > 0 \quad \text{and} \quad y'\left(\frac{\pi}{2}\right) < 0.$$ □

LEMMA 2.4. *for $-1 < b$ the boundary value problem*

(2.4)
$$y^{(4)} + by^+ = 1 + k\cos x \quad in \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$y\left(\pm\frac{\pi}{2}\right) = y''\left(\pm\frac{\pi}{2}\right) = 0$$

has a unique solution.

Proof. The solution y of (2.2) is positive, hence it is also a solution of (2.4). Uniqueness follows from the contraction mapping principle in the following familiar way. The eigenvalues of $My = \lambda y$, where $M = D^4$, with the boundary conditions is given in (2.2), are all ≥ 1. Hence for any $c < 1$,

$$\|(M-c)^{-1}\| = \frac{1}{1-c}.$$

Any problem $My = f(y,x)$ with $c \leq f_y \leq 1-\varepsilon$ has a unique solution, since solutions y are characterized by

$$y = (M-c)^{-1}[f(y,x) - cy],$$

where the right hand side is Lipschitz continuous with a Lipschitz constant $\leq \dfrac{1-\varepsilon-c}{1-c} < 1$. □

The following lemma is the final step in the proof of Theorem 2.1.

LEMMA 2.6. *Let* $4 < b < 19$. *Then there exists* $\gamma > 0$, $\varepsilon_0 > 0$ *such that*

$$d_{LS}(u - L^{-1}(1 + k\cos x - bu^+ + \varepsilon h), B_\gamma(y), 0) = -1$$

for $|\varepsilon| < \varepsilon_0$ *where* $k \geq 0$ *and* y *is the unique solution of (2.4).*

Now we prove our main result, Theorem 2.1.

Proof of Theorem 2.1. Equation (1.4) can be written in the form

$$Su := u - L^{-1}(1 + k\cos x - bu^+ + \varepsilon h) = 0.$$

The degree of Su on a large ball of radius $R > R_0$ is $+1$ by Lemma 2.1. We know from Lemma 2.6 that the degree on the ball $B_\gamma(y)$ is -1. Choosing $R > R_0$ so large that $B_R \subset B_\gamma(y)$, we can conclude that

$$d_{LS}(Su, B_R - B_\gamma(y), 0) = 2.$$

Therefore, the equation (1.4) has at least two solution, one in $B_\gamma(y)$ and the other one in $B_R - B_\gamma(y)$. This conclude the proof of Theorem 2.1. □

3. The Suspension Bridge Equation under a Weak Condition.

In this section we investigate solutions of a nonlinear suspension bridge equation under weak even condition

(3.1)

$$-\frac{1}{4}u_{ttxx} + u_{tt} + u_{xxxx} + bu^+ = 1 + \varepsilon h(x,t) \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R,$$

$$u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0,$$

$$u \text{ is } \pi\text{-periodic in } t \text{ and even in } x.$$

Let L be the differential operator

$$Lu = -\frac{1}{4}u_{ttxx} + u_{tt} + u_{xxxx}.$$

The eigenvalue problem for $u(x,t)$

$$Lu = \lambda u \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times R, \tag{3.2}$$

$$u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0,$$

$$u(x,t) = u(-x,t) = u(x, t+\pi)$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - m^2((2n+1)^2 + 4) \qquad (m, n = 0, 1, 2, \dots)$$

and corresponding normalized eigenfunction ϕ_{mn}, ψ_{mn} $(m, n \geq 0)$ given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi}\cos(2n+1)x \qquad \text{for} \quad n \geq 0,$$

$$\phi_{mn} = \frac{2}{\pi}\cos 2mt\cos(2n+1)x \qquad \text{for} \quad m > 0, n \geq 0,$$

$$\psi_{mn} = \frac{2}{\pi}\cos 2mt\cos(2n+1)x \qquad \text{for} \quad m > 0, n \geq 0.$$

We note that all eigenvalue in the interval $(-36, 29)$ are given by

$$\lambda_{20} = -19 < \lambda_{10} = -4 < \lambda_{00} = 1.$$

Let $\mathbb{Q}$ be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and H the Hilbert space defined by

$$H_0 = \{u \in L^2(\mathbb{Q}) : u \text{ is even in } x\}.$$

We define a subspace H of H_0 as follows;

$$\{u \in H_0 : u = \sum (h_{mn}\phi_{mn} + \tilde{h}_{mn}\psi_{mn}), \quad \sum |\lambda_{mn}|(h_{mn}^2 + \tilde{h}_{mn}^2) < \infty\}$$

with a norm

$$|||u||| = \left[\sum |\lambda_{mn}|(h_{mn}^2 + \tilde{h}_{mn}^2)\right]^{\frac{1}{2}}.$$

Then this normed space is complete. The set of function $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal base in H.

Let V be the 2-dimensional subspace of H which is the closure of the span of the functions ϕ_{10} and ψ_{10}, both of which have the same eigenvalue $\lambda_{10} = -4$. Let W be the orthogonal complement of V in H.

We first consider the uniqueness theorem when $-1 < b < 4$.

THEOREM 3.1. *Let $\|h\| = 1$ and $-1 < b < 4$. Then for small $\varepsilon > 0$ the equation*

$$Lu + bu^+ = 1 + \varepsilon h \quad in \quad H \tag{3.3}$$

has a unique solution.

Proof. Let $\|h\| = 1$ and $-1 < b < 4$. Let $\delta = \frac{3}{2}$. The equation (3.3) is equivalent to

$$u = (L + \delta)^{-1}[-(b - \delta)u^+ - \delta u^- + 1 + \varepsilon h(x, t)], \tag{3.4}$$

where $(L + \delta)^{-1}$ is a compact, self-adjoint, linear map from H into H with norm $\frac{1}{2}$. We note that

$$\|(b-\delta)(u_2^+ - u_1^+) + \delta(u_2^- - u_1^-)\| \leq \max\{|b-\delta|, \delta\}\|u_2 - u_1\| < 2\|u_2 - u_1\|.$$

It follows that the right hand side of (3.4) defines a Lipschitz mapping of H into H with Lipschitz constant $\gamma < \frac{1}{2} \times 2 = 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in H$ of (3.4). □

The main theorem in this section is the following.

THEOREM 3.2. *Let $h \in W$, $\|h\| = 1$, and $4 < b < 19$. Then there exists $\varepsilon_0 > 0$ depending on b and h such that if $|\varepsilon| < \varepsilon_0$ equation (3.3) has at least three solutions.*

Let us define the functional $I_b(u)$ on H, corresponding to equation (3.3), as follows;

$$I_b(u)$$
$$= \int_Q \left[\frac{1}{2}\left(-|u_t|^2 - \frac{1}{4}|u_{tx}|^2 + |u_{xx}|^2\right) + \frac{b}{2}|u^+|^2 - u - \varepsilon h(x,t)u\right] dtdx.$$

Then I_b is continuous and Fréchet differentiable in H.

The solutions of (3.3) coincide with the critical points of I_b.

LEMMA 3.1. *Let $4 < b < 19$, $h \in W$ with $|||h||| = 1$, and let $v \in V$ be given. Then for small $\varepsilon > 0$, there exists a unique solution $z \in W$ of the equation*

$$Lz + (I-P)[b(v+z)^+ - 1 - \varepsilon h(x,t)] = 0 \quad in \quad W.$$

If $z = \theta(v)$, then θ is continuous on V and we have $DI_b(v+\theta(v))(w) = 0$ for all $w \in W$. If $\tilde{I}_b : V \longrightarrow R$ is defined by $\tilde{I}_b(v) = I_b(v + \theta(v))$, then $\tilde{I}_b$ has a contiunous Fréchet derivative $D\tilde{I}_b$ with respect to v and

$$D\tilde{I}_b(v)(h) = DI_b(v+\theta(v))(h) \quad for\ all \quad h \in V.$$

If v_0 is a critical point of $\tilde{I}_b$, then $v_0 + \theta(v_0)$ is a solution of (1.4) and conversely every solution of (1.4) is of this form. In particular $\theta(v)$ satisfies a uniform Lipschitz condition in v with respect to the $L^2(Q)$ norm (also the norm $|||\cdot|||$).

The method of the proof of Theorem 3.2 is reduced to the problem in an infinite dimensional Hilbert space to an equivalent finite-dimensional one via a variational reduction method (Lemma 3.1). These methods were first used in [4], [8] and were afterwards extended in [2], to the case we with to use. For the details of the proof of this theorem, we can refer [6].

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