Kangweon-Kyungki Math. Jour. 3 (1995), No. 1, pp. 89-91

SOME PROPERTIES OF $*$–BARRELLEDNESS

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1. Introduction.

S. G. Gayal and K. Anjaneyulu([1],[2]) introduced the concepts of two new classes of locally convex spaces, which they call $*$-barrelled and quasi$*$-barrelled spaces to generalize the well known classes of barrelled and quasibarrelled spaces respectively. In this note, we consider a relationship between quasi $*$-barrelled spaces and semi-Montel spaces and equivalence of barrelledness, quasibarrelledness and $*$-barrelledness of reflexive locally convex spaces. Also we show the following fact: Let E be a locally convex space and F a reflexive locally convex space. Suppose that there exists a continuous linear almost open mapping f of E into F. If E is a quasi $*$-barrelled space, so is F. Let E be a locally convex space and E' its dual space. A subset of E is said to be a $*$-barrel(bornivorous $*$-barrel) if it is the polar of a relatively compact subset of E' for the topology $\sigma(E', E)(\beta(E', E))$([1],[2]). The locally convex space E is said to be $*$-barrelled(quasi $*$-barrelled) if every $*$-barrel(bornivorous $*$-barrel) in E is a neighborhood of 0 ([1],[2]). It is well known ([1],[2])that a locally convex space E is $*$-barrelled(quasi $*$-barrelled) if and only if every subset of E' which is relatively $\sigma(E', E)(\beta(E', E))$-compact is equicontinuous. Every barrelled space is quasibarrelled and $*$-barrelled; and quasibarrelled($*$-barrelled) space is quasi $*$-barrelled. All spaces in this note are to be Hausdorff. The notations and definitions used here, and in what follows, are those of [3], unless explicitly stated to the contrary.

2. Results.

Received October 28, 1994.

THEOREM 1. *If E is a quasicomplete quasi $*$-barrelled locally convex space, then it is semi-Montel.*

Proof. This follows directly from proposition 1[1] and proposition 11.5.2 [4].

THEOREM 2. *Let E be a reflexive locally convex space. Then the following statements are equivalent:*

(1) *E is barrelled*
(2) *E is quasibarrelled*
(3) *E is $*$-barrelled.*

Proof. (1) $\Longrightarrow$ (2) is obvious. (2) $\Longrightarrow$ (3): Let B be a relatively $\sigma(E', E)$-compact subset of E'. Then it is $\sigma(E', E)$-bounded. Since E is quasicomplete, B is $\beta(E', E)$-bounded. Since E is quasibarrelled, B is equicontinuous. Hence E is $*$-barrelled. (3) $\Longrightarrow$ (1): Let B be a $\sigma(E', E)$-bounded subset of E'. Since E' is a semireflexive locally convex space, B is a relatively $\sigma(E', E)$-compact subset of E'. Since E is a $*$-barrelled space, B is an equicontinuous subset of E'. Hence E is a barrelled space. □

LEMMA. *Let E and F be locally convex spaces and f a continuous linear mapping of E into F. If B is any bornivorous $*$-barrel in F, Then $f^{-1}(B)$ is also a bornivorous $*$-barrel in E.*

Proof. Since $f : E \to F$ is continuous, its transpose $f' : F' \to E'$ is continuous for $\sigma(F', F)$ and $\sigma(E', E)$ and also for $\beta(F', F)$ and $\beta(E', E)$. Let B be a bornivorous $*$-barrel in F.Then there is a relatively $\beta(F', F)$-compact subset M of F' such that $B = M^o$. Since $f'(\overline{M})$ is a $\beta(E', E)$-compact subset of E',

$$f'(\overline{M}) \subset \overline{f'(M)} \subset \overline{f'(\overline{M})} = f'(\overline{M})$$

and $\overline{f'(M)} = f'(\overline{M})$. Therefore $f'(M)$ is a relatively $\beta(E', E)$-compact subset of E'. And $(f'(M))^o = \{(f')'\}^{-1}(M^o) = f^{-1}(B)$. Hence $f^{-1}(B)$ is a bornivorous $*$-barrel in E.

THEOREM 3. *Let E be a locally convex space and F a reflexive locally convex space. Suppose that there exists a continuous linear* *almost open mapping f of E into F.If E is a quasi $*$-barrelled space, so is F.*

Proof. Let B be a bornivorous $*$-barrel in F. Then $f^{-1}(B)$ is also a bornivorous $*$-barrel in E by Lemma. Since E is a quasi $*$-barrelled space, it follows that $f^{-1}(B)$ is a neighborhood of 0 in E. Since f is an almost open mapping of E into F and the topology $\beta(F, F')$ on F is compatible with the duality between F and F' by assumption,

$$\overline{f(f^{-1}(B))} \subset \overline{B} = B.$$

Hence B is a neighborhood of 0 in F. Therefore F is a quasi $*$-barrelled space.

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