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DERIVATIONS OF PRIME RINGS AND BANACH ALGEBRAS

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1. Introduction.

Throughout, R represents an associative ring with center $Z(R)$. We shall write char(R) for the characteristic of ring R. We write $[x,y]$ for $xy - yx$, and $< x,y >$ for $xy + yx$. Then we have $[< y,x >, x] = [< x,y >, y]$ for all $x, y \in R$. Let rad(A) denote the (Jacobson) radical of an algebra A. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A mapping F from R to R is said to be commuting on R if $[F(x), x] = 0$ holds for all $x \in R$. In Theorem 1 in Brešar and Vukman [2], they prove the following: Let D and G be continuous derivations of a Banach algebra A such that $[D^2(x) + G(x), x] \in rad(A)$ for all $x \in A$. Then both D and G map A into rad(A). In Vukman [8] he has proved that in case there exists a nonzero derivation $D : R \longrightarrow R$, where R is a prime ring of characteristic different from 2 , 3, such that the mapping $x \mapsto [D(x), x]$ is commuting on R, R is commutative. We are to prove that if there is a derivation D on a noncommutative prime ring R such that $[< Dx, x >, x] = 0$ for all $x \in R$ and char$(R) \neq 2$, then we have $D = 0$. And making use of the result, we can prove that the range of a continuous linear Jordan derivation on a noncommutative Banach algebra A is contained in the radical under the condition $[< Dx, x >, x] \in rad(A)$ for all $x \in A$. For further references we refer to [1,3,4,9].

2. Main Results.

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THEOREM 2.1. *Let R be a noncommutative prime ring with* char$(R) \neq 2$. *Suppose there exists a derivation $D: R \longrightarrow R$ such that $x \mapsto < Dx, x >$ is commuting on R. Then we have $D = 0$.*

Proof. We define a mapping $B(\cdot,\cdot) : R \times R \longrightarrow R$ by

$$B(x, y) = [Dx, y] + [Dy, x], \quad x, y \in R.$$

$B(y, x) = B(x, y)$ for all $x, y \in R$ and additive in both arguments. We have the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + (Dx)[y, z] + [x, z]Dy,$$

for all $x, y, z \in R$. We also define the following: $e(x) = 2Dx, f(x) = [e(x), x], x \in R$. Then $f(x) = B(x, x), \ \ x \in R$. And it follows that

$$f(x + y) = f(x) + f(y) + 2B(x, y), \quad x, y \in R.$$

The assumption of the theorem can now be written in the form

$$[< Dx, x >, x] = 0, \quad x, y \in R. \tag{1}$$

From (1), one obtains

$$f(x)x + xf(x) = 0, \ x \in R. \tag{2}$$

The linearization of (2) gives

$$\begin{aligned} &f(x)y + yf(x) + 2B(x, y)x + 2xB(x, y) + (f(y)x + xf(y) \\ &+ 2B(x, y)y + 2yB(x, y)) = 0, \quad x, y \in R. \end{aligned} \tag{3}$$

Replacing $-y$ for y in (3), we have

$$\begin{aligned} &- f(x)y - yf(x) - 2B(x, y)x - 2xB(x, y) + (f(y)x + xf(y) \\ &+ 2B(x, y)y + 2yB(x, y)) = 0, \quad x, y \in R. \end{aligned} \tag{4}$$

Subtracting (4) from (3), using the condition char$(R) \neq 2$ we obtain

$$f(x)y + yf(x) + 2B(x, y)x + 2xB(x, y) = 0, \quad x, y \in R. \tag{5}$$

Substituting yx for y in (5), it follows that

$$f(x)yx + yf(x)x + 2B(x,y)x^2 + 2xB(x,y)x + yf(x)x + yxf(x) + 2xyf(x) + 2xyf(x) + [y,x]e(x)x + x[y,x]e(x) = 0, \quad x, y \in R. \tag{6}$$

Combining (2), (5) with (6), we arrive at

$$2xyf(x) + [y,x]e(x)x + x[y,x]e(x) = 0, \quad x, y \in R. \tag{7}$$

Replacing $e(x)y$ for y in (7), one obtains

$$2xe(x)yf(x) + e(x)[y,x]e(x)x + f(x)ye(x)x + xe(x)[y,x]e(x) + xf(x)ye(x) = 0, \quad x, y \in R. \tag{8}$$

Left multiplication of (7) by $e(x)$ leads to

$$2e(x)xyf(x) + e(x)[y,x]e(x)x + e(x)x[y,x]e(x) = 0, \quad x, y \in R. \tag{9}$$

Subtracting (8) from (9), we arrive at

$$2f(x)yf(x) - f(x)ye(x)x + f(x)[y,x]e(x) - xf(x)ye(x) = 0, \quad x, y \in R. \tag{10}$$

Since $f(x)[y,x]e(x) = f(x)yxe(x) - f(x)xye(x), x, y \in R$, (10) reduces to

$$f(x)yf(x) - (f(x)x + xf(x))ye(x) = 0, \quad x, y \in R. \tag{11}$$

Hence, combining (2) with (11) we arrive at

$$f(x)yf(x) = 0, \quad x, y \in R. \tag{12}$$

By primeness of R it follows from (12) that

$$f(x) = 0, \quad x \in R.$$

Therefore, by Lemma 3 in [6] we have

$$Dx = 0, \quad x \in R.$$

The proof of the theorem is complete. □

THEOREM2.2. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that $[< Dx, x >, x] \in rad(A), x \in A$. Then we have $DA \subseteq rad(A)$.*

Proof. Let $D : A \longrightarrow A$ be a continuous linear Jordan derivation. Then by Lemma 3.2 in [7], $DP \subseteq P$. Hence we have $D(rad(A)) \subseteq rad(A)$. And so, we may assume that A is semsimple. Now since $DP \subseteq P$, we can introduce a Jordan derivation $D_P : A/P \longrightarrow A/P$ by $D_P(x + P) = Dx + P$, $x \in A$. When P is a primitive ideal, A/P is a prime algebra. Thus by Herstein's result D_P is a derivation. And the assumption of the theorem gives $[< D_P(x + P), x + P >, x + P] = 0, x \in A$. Hence in case A/P is noncommutative, we have $D_P = 0$ by Theorem 2.1. It is sufficient to prove that $D_P = 0$ when A/P is commutative. Johnson [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. By this result and Singer-Wermer's Theorem, we obtain that there is the only zero derivation on commutative Banach algebras. Hence in this case we have $D_P = 0$ as well. Consequently, this means that $DA \subseteq rad(A)$. The proof of the theorem is complete. □

COROLLARY 2.3. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that $[D(x^2), y] \in rad(A), x, y \in A$. Then we have $DA \subseteq rad(A)$.*

Proof. All the assumptions of Theorem 2.2 are fulfilled. □

THEOREM 2.4. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that $D(x^2) \in Z(A), x \in A$. Then we have $D = 0$.*

Proof. The proof goes through in the same fashion as the proof of Theorem 2.2 except the fact that at the beginning of the proof we have to use the fact that any linear Jordan derivation on a semisimple Banch algbera is continuous(see [1, Theorem 6]). □

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