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SOME NOTES ON THE EXTENSION OF B–VALUED INNER PRODUCT

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1. Introduction.

B-valued inner product has been studied by Paschke([1],[4],[5],[6]), Arveson. It is different from inner product in that codomain is a C^*-algebra and its axioms are compatible with module action.

In particular, Paschke investigated the dual space X' which is composed of bounded module maps of pre-Hilbert B-module X into a C^*-algebra B(this has similar properties with dual space of a Banach space). He has lifted the B-valued inner product on a pre-Hilbert space X to a B-valued inner product on X' and connected with representation theory with respect to completely positive map([5],[6]).

In this setting, there are two ways of norming X', as bounded operators from X into B, and inner product norm $\|\cdot\|_{X'}$ on the other. In fact, these norms are identical([5],[Corollary 2.8]). Also we can conjecture problems that the B-valued inner product on X can be lifted to a B-valued inner product on X''(the bidual of X).

After appropriate identification, we can regard X as a submodule of X''([Remark 2]), and this note is the investigation of the above conjecture and structural relations of $X, X', X'', (X'')'$([Lemma 3.2], [Theorem 3.6], [Theorem 3.7]).

2. B-valued inner product and its Extension to X'.

Let B be a C^*-algebra and X a right B-module. We will denote the action of an element $b \in B$ on $x \in X$ by $x \cdot b$; it is assumed that X has a vector space structure compatible with that of B in the sense that $\lambda(x \cdot b) = (\lambda x) \cdot b = x \cdot (\lambda b)$ for all $x \in X, b \in B, \lambda \in C$.

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DEFINITION 2.1. A pre-Hilbert B-module is a right B-module X equipped with a conjugate-bilinear map $\langle\cdot,\cdot\rangle : X \times X \longrightarrow B$ satisfying :

(1) $\langle x, x\rangle \geq 0 \quad \forall x \in X$;
(2) $\langle x, x\rangle = 0 \quad$ only if $x = 0$;
(3) $\langle x, y\rangle^{*} = \langle y, x\rangle \quad \forall x, y \in X$;
(4) $\langle x \cdot b, y\rangle = \langle x, y\rangle b \quad \forall x, y \in X, \quad b \in B.$

The map $\langle\cdot,\cdot\rangle$ will be called a *B-valued inner product* on X.

EXAMPLE 2.2. If H is a Hilbert space, then the algebraic tensor product $H \odot B$ becomes a pre-Hilbert B-module. For, defining by $(\xi \otimes a, b) \rightarrow \xi \odot ab$, then $H \odot B$ becomes a right B-module.

Define $\langle\cdot,\cdot\rangle : H \odot B \times H \odot B \longrightarrow B$ by $\langle \xi \odot a, \eta \odot b\rangle = (\xi, \eta) b^{\star} a$. Then $(\xi, \xi) a^{\star} a \geq 0$, and $(\xi, \xi) a^{\star} a = 0 \Leftrightarrow (\xi, \xi) = 0$ or $a^{\star} a = 0 \Leftrightarrow \xi = 0$ or $a = 0$.

Also, $\langle \xi \otimes a, \eta \otimes b\rangle^{*} = (\xi, \eta)^{*}(b^{\star} a)^{\star} = (\eta, \xi) a^{\star} b = \langle \eta \otimes b, \xi \odot a\rangle$.
$\langle(\eta \otimes b) \cdot c, \xi \otimes a\rangle = \langle \eta \odot bc, \xi \otimes a\rangle = (\eta, \xi) a^{*}(bc) = (\eta, \xi)(a^{*} b) c = \langle \eta \otimes b, \xi \otimes a\rangle c.$

For a pre-Hilbert B-module X, define $\| \cdot \|_X$ on X by $\| x \|_X = \| \langle x, x\rangle \|^{1/2}$.

PROPOSITION 2.3([5],[6]). *$\| \cdot \|_X$ is a norm on X and satisfies:*
(1) $\| x \cdot b \|_X \leq \| x \|_X \| b \| \quad \forall x, \in X, \quad b \in B$;
(2) $\langle y, x\rangle\langle x, y\rangle \leq \| y \|_X^2 \langle x, x\rangle \quad \forall x, y \in X$;
(3) $\| \langle x, y\rangle \| \leq \| x \|_X \| y \|_X \quad \forall x, y \in X$.

REMARK 1. Because of (1), X is a normed B-module, and a pre-Hilbert B-module X which is complete with respect to $\| \cdot \|$ will be called a ***Hilbert B-module.*** For a Hilbert B-module X, we let X' denote the set of all bounded B-module maps(i.e, B-linear maps) of X into B. Then X' becomes a vector space if we define scalar multiplication on X' by $(\lambda\tau)(x) = \overline{\lambda}\tau(x) (\tau \in X', x \in X, \lambda \in C)$ and addition maps in X' pointwise. Also,

X' becomes a right B-module if we set $(\tau \cdot b)(x) = b^{\star}\tau(x)$ for $\tau \in X', b \in B, x \in X$.

PROPOSITION 2.4([5]). *Let X,Y be pre-Hilbert B-modules. For a linear map $T : X \longrightarrow Y$, the following are equivalent: (1). T is* *bounded and* $T(x \cdot b) = (Tx) \cdot b \quad \forall x \in X, b \in B$. *(2). There is a real number* $K \geq 0$ *such that* $\langle Tx, Tx\rangle \leq K\langle x, x\rangle \quad \forall x \in X$.

REMARK 2. From the above proposition, for a bounded B-module map T,

$\| T \| = inf\{K^{1/2} : \langle Tx, Tx\rangle \leq K\langle x, x\rangle \quad \forall x \in X\}$ and X' is precisely the set of linear maps $\tau : X \longrightarrow B$ such that for some real $K \geq 0, \quad \tau^{\star}\tau(x) \leq K\langle x, x\rangle \quad \forall x \in X$. Each $x \in X$ gives rise to a map $\hat{x} \in X'$ defined by $\hat{x}(y) = \langle y, x\rangle (y \in X)$([Proposition 2.3]). The map $x \rightarrow \hat{x}$ is an isometric module map of X into X'. We may thus regard X as a submodule of X' by identify X with $\hat{X}$.

THEOREM 2.5([5]). *Let* X *be a pre-Hilbert* B*-module. The* B*-valued inner product* $\langle \cdot, \cdot\rangle$ *extends to* $X' \times X'$ *in such a way as to make* X' *into a Hilbert B-module.*

By the above theorem, since X' becomes a pre-Hilbert B-module, we can consider the following example:

EXAMPLE 2.6. Consider the right B-module $B \times X$ for any pre-Hilbert B-module X. Take $\tau \in X'(\tau \neq 0)$ and $t > \| \tau \|_{X'}$. Define $[\cdot, \cdot]_{\tau,t} : (B \times X) \times (B \times X) \longrightarrow B$ by $[(a, x), (b, y)]_{\tau,t} = t^2 b^{\star} a + b^{\star}\tau(x) + \tau(y)^{\star} a + \langle x, y\rangle$. Then

$$\begin{aligned}
[(a, x), (b, y)]_{\tau,t} &= t^2 b^{\star} a + \tau(y)^{\star} a + b^{\star}\tau(x) + \langle x, y\rangle \\
&= [a^{\star} b + a^{\star}\tau(y) + \tau(x)^{\star} b + \langle y, x\rangle]^{\star} \\
&= [(b, y), (a, x)]^{\star}_{\tau,t},
\end{aligned}$$

$$\begin{aligned}
[(a, x) \cdot k, (b, y)]_{\tau,t} &= [(ak, x \cdot k), (b, y)] \\
&= t^2 b^{\star}(ak) + b^{\star}\tau(x \cdot k) + \tau(y)^{\star}(ak) + \langle x \cdot k, y\rangle \\
&= t^2(b^{\star} a)k + b^{\star}\tau(x)k + \tau(y)^{\star} ak + \langle x, y\rangle k \\
&= [(a, x), (b, y)]k.
\end{aligned}$$

Taking $(a,x) \in B \times X$, then,

$$\begin{aligned}[(a,x),(a,x)]_{\tau,t} &= t^2 a^\star a + a^\star \tau(x) + \tau(x)^\star a + \langle x, x\rangle \\ &\geq t^2 a^\star a + a^\star \tau(x) + \tau(x)^\star a + \| \tau \|_{X'}^{-2} \tau(x)^\star \tau(x) \\ &= t^2 a^\star a + a^\star \tau(x) + \tau(x)^\star a + t^{-2}\tau(x)^\star \tau(x) \\ &= \left(ta + t^{-1}\tau(x)\right)^\star \left(ta + t^{-1}\tau(x)\right) \geq 0.\end{aligned}$$

If $[(a,x),(a,x)]_{\tau,t} = 0$,

$$\begin{aligned}[(a,x),(a,x)]_{\tau,t} &= t^2 a^\star a + a^\star \tau(x)^\star a + \langle x, x\rangle \\ &= \left(ta + t^{-1}\tau(x)\right)^\star \left(ta + t^{-1}\tau(x)\right) = 0\end{aligned}$$

(i.e. equality holds in each above step). In particular, $\left(\| \tau \|_{X'}^{-2} - t^{-2}\right)\tau(x)^\star \tau(x) = 0$, and so $(a,x) = (0,0)$, thus $[\cdot,\cdot]_{\tau,t}$ is a B-valued inner product on $B \times X$.

LEMMA 2.7. *Let $\| \cdot \|_{\tau,t}$ be a norm on $B \times X$ gotten from the above inner product. Then $\| \tau \cdot b + \hat{y} \| \leq \| (b,y) \|_{\tau,t} \quad \forall x \in X, \ \ b \in B$.*

Proof. For all $x \in X$,

$$\| (0,x) \|_{\tau,t} = \| [(0,x),(0,x)] \|^{1/2} = \| \langle x,x\rangle \|^{1/2} = \| x \|_X.$$

For $x, y \in X, \quad b \in B$, we have

$$\begin{aligned}\| (\tau \cdot b + \hat{y})(x) \| &= \| b^\star \tau(x) + \langle x, y\rangle \| \\ &= \| [(0,x),(b,y)]_{\tau,t} \| \leq \| (0,x) \|_{\tau,t} \cdot \| (b,y) \|_{\tau,t} \\ &= \| x \|_X \| (b,y) \|_{\tau,t} ([Proposition\ \ 2.3]).\end{aligned}$$

□

COROLLARY 2.8([5]). *The operator norm and inner product norm in X' are identical.*

Proof. By the above theorem, X' is a Hilbert B-module. Letting $\| \cdot \|_{X'}$ denote the operator norm on X', we have, for $\tau \in X'$ and $x \in X$, $\tau(x)^\star \tau(x) = \langle \tau, \hat{x}\rangle \langle \hat{x}, \tau\rangle \leq \| \tau \|_{X'}^2 \langle x, x\rangle$([Proposition 2.3]), therefore $\| \tau \| \leq \| \tau \|_{X'}$ ([Remark 2]). On the other hand, $\| \tau \|_{X'}^2 \leq \| \tau \|^2$, forcing $\| \tau \|_{X'} = \| \tau \|$.

3. The extension of B-valued inner product to X''.

LEMMA 3.1. *If X is a normed B-module, there exists an module map of X into X''.*

Proof. For $x \in X$, define $\phi : X \longrightarrow X''(x \rightarrow \dot{x})$ by $\dot{x}(\tau) = \tau(x)^\star(\tau \in X')$. Then $\dot{x}(\tau \cdot b) = [(\tau \cdot b)x]^\star = [b^\star\tau(x)]^\star = \tau(x)^\star b = \dot{x}(\tau)b$, $\| \dot{x}(\tau) \| = \| \tau(x)^\star \| = \| \tau(x) \| \leq \infty$,

and so $\dot{x} \in X''$. Also, $(\hat{x} \cdot b)\dot{}(\tau) = \tau(x \cdot b)^\star = [\tau(x)b]^\star = b^\star\tau(x)^\star = b^\star\hat{x}(\tau) = (\hat{x} \cdot b)(\tau)$.

Thus $\phi(x \cdot b) = \phi(x) \cdot b$ (i.e. module map).

LEMMA 3.2. *If X is a normed B-module, then there is a bounded module map of X'' into X'.*

Proof. For $\Gamma \in X''$, define $\tilde{\Gamma}$ on X' by $\tilde{\Gamma}(x) = \Gamma(\hat{x})(x \in X)$, then

$$\tilde{\Gamma}(x \cdot b) = \Gamma[(x \hat{\cdot} b)] = \Gamma(\hat{x} \cdot b) = \Gamma(\hat{x}) \cdot b = \tilde{\Gamma}(x) \cdot b(\ i.e\ \ \tilde{\Gamma} \in X').$$

Now define a map $\Psi : X'' \longrightarrow X'(\Gamma \rightarrow \tilde{\Gamma})$, then since

$$\begin{aligned}(\Gamma_1 \tilde{+} \Gamma_2)(x) &= (\Gamma_1 + \Gamma_2)(\hat{x}) \\ &= \Gamma_1(\hat{x}) + \Gamma_2(\hat{x}) = \tilde{\Gamma}_1(x) + \tilde{\Gamma}_2(x)\end{aligned}$$

and

$$\begin{aligned}(\Gamma \tilde{\cdot} b)(x) &= (\Gamma \cdot b)(\hat{x}) \\ &= \Gamma(\hat{x}) \cdot b = \tilde{\Gamma}(x) \cdot b,\end{aligned}$$

$\| \tilde{\Gamma}(x) \|_B = \| \Gamma(\hat{x}) \|_B \leq \| \Gamma \|_{X''} \cdot \| \hat{x} \|$.

Thus Ψ is a bounded module map(in fact, Ψ is an isometry[Lemma 3.4]). □

Now we introduce the concrete extension of B-valued inner product on X to X''.

As a previous statement, the method of extension is similar to that of Paschke's.

Define $\langle \cdot, \cdot \rangle : X'' \times X'' \longrightarrow B$ by $\langle \Gamma, \Phi \rangle = \Phi(\tilde{\Gamma})$. Then it is clear that this map is conjugate bilinear and will be a B-valued inner product on X''([Lemma 3.5],[Theorem 3.6]). For $x, y \in X$, $\langle \dot{x}, \dot{y} \rangle = \dot{y}\Big((\tilde{\dot{x}})\Big) = \dot{y}(\hat{x}) = \hat{x}(y)^\star = \langle y, x \rangle^\star = \langle x, y \rangle$.

So $\langle \cdot, \cdot \rangle$ is an extension of the original inner product on X.

LEMMA 3.3. *Let Y be a submodule of X' containing $\hat{X}$. For any $F \in Y'$, we have $\| F \|_{Y'} = \| F|_X \|$.*

Proof. We may assume without loss of generality that $\| F \|_{Y'} = 1$. Define $\tau \in X'$ by $\tau(x) = F(\hat{x})(x \in X)$. We have $\| \tau \|_{X'} \leq 1$ and must establish the reverse inequality. Take $\psi \in Y$ with $\| \psi \|_{X'} < 1$ and set $c = F(\psi)$. For brevity, let $[\cdot,\cdot]$ denote the B-valued inner product $[\cdot,\cdot]_{\psi,1}$ on $B \times X$ defined in Example 2.6 and let $\| \cdot \|$ be the corresponding norm on $B \times X$. For $a \in B, x \in X$, we have, using Lemma 2.7,

$$\| ca + \tau(x) \| = \| F(\psi \cdot a + \hat{x}) \| \leq \| \psi \cdot a + \hat{x} \|_{X'} \leq \| (a, x) \|,$$

so the map $(a, x) \to ca + \tau(x)$ of $B \times X$ into B is a bounded module map of norm ≤ 1 with respect to the inner product $[\cdot,\cdot]$. By Remark 2, we have $(ca + \tau(x))^*(ca + \tau(x)) \leq [(a, x), (a, x)]$ for all $a \in B, x \in X$. That is,

$$a^*c^*ca + a^*c^*\tau(x) + \tau(x)^*ca + \tau(x)^*\tau(x) \leq a^*a + a^*\psi(x) + \psi(x)^*a + \langle x, x\rangle.$$

Setting $a = -2\psi(x)$ and collecting terms, we obtain, for all $x \in X$.

$$4\psi(x)^*c^*c\psi(x) + \tau(x)^*\tau(x) \leq \langle x, x\rangle + 2(\psi(x)^*c^*\tau(x) + \tau(x)^*c\psi(x)).$$

But $\psi(x)^*c^*\tau(x) + \tau(x)^*c\tau(x) \leq \tau(x)^*c^*c\tau(x) + \tau(x)^*\tau(x)$,

$$2\psi(x)^*c^*c\psi(x) \leq \langle x, x\rangle + \tau(x)^*\tau(x) \leq (1 + \| \tau \|_{X'}^2)\langle x, x\rangle \quad \forall x \in X.$$

Hence $\| \psi \cdot c^* \|_{X'} \leq 2^{-1/2}(1 + \| \tau \|_{X'}^2)^{1/2}$ and consequently, using $\| F \|_{Y'} = 1$,

$$\begin{aligned} \| F(\psi \cdot c^*) \| &= \| cc^* \| = \| c \|^2 \leq \| F \|_{Y'} \| \psi \cdot c^* \|_{X'} \\ &= \| \psi \cdot c^* \|_{X'} \leq 2^{-1/2}(1 + \| \tau \|_{X'}^2)^{1/2}. \end{aligned}$$

This holds for any $\psi \in Y$ with $\| \psi \|_{X'} < 1$; since $\| F \|_{Y'} = 1$, we must therefore have $1 \leq 2^{-1/2}(1 + \| \tau \|_{X'}^2)^{1/2}$, which forces $\| \tau \|_{X'} \geq 1$. This completes the proof. □

LEMMA 3.4. *The map Ψ in Lemma 3.2 is an isometry.*

Proof. For any $F \in X''$, $\| F \|_{X''} = \| F|_X \|$ ($Y = X'$ in Lemma 3.3). Since $\tilde{\Gamma}(x) = \Gamma(\hat{x})(x \in X)$ for $\Gamma \in X''$, $\| \Gamma \|_{X''} = \| \Gamma|_X \| = \| \tilde{\Gamma} \|_{X'}$. □

By elementary calculation, we can get the following Lemma.

LEMMA 3.5. $\langle \Gamma, \Gamma \rangle \geq 0$ *and* $\| \langle \Gamma, \Gamma \rangle \| = \| \Gamma \|^2_{X''}$ *for all* $\Gamma \in X''$.

THEOREM 3.6. *Let X be a pre-Hilbert B-module. Then the B-valued inner product $\langle \cdot, \cdot \rangle$ on X extends to $X'' \times X''$ in such a way as to make X'' into a Hilbert B-module.*

Proof.

$$\langle \Gamma \cdot b, \Phi \rangle = \Phi\Big((\Gamma \tilde{\cdot} b)\Big) = \Phi(\tilde{\Gamma} \cdot b)$$
$$= \Phi(\tilde{\Gamma})b = \langle \Gamma, \Phi \rangle b$$

Also, $\langle \Gamma + \Phi, \Gamma + \Phi \rangle \geq 0$, $\quad \langle \Gamma + i\Phi, \Gamma + i\Phi \rangle \geq 0$ ([Lemma 3.5]).

$$\langle \Gamma + \Phi, \Gamma + \Phi \rangle = \langle \Gamma + \Phi, \Gamma + \Phi \rangle^*$$
$$= \langle \Gamma, \Gamma \rangle + \langle \Gamma, \Phi \rangle^* + \langle \Phi, \Gamma \rangle^* + \langle \Phi, \Phi \rangle.$$

Thus

$$\langle \Gamma, \Phi \rangle + \langle \Phi, \Gamma \rangle = \langle \Gamma, \Phi \rangle^* + \langle \Phi, \Gamma \rangle^* \quad (\star),$$
$$\langle \Gamma, \Phi \rangle - \langle \Phi, \Gamma \rangle = -\langle \Gamma, \Phi \rangle^* + \langle \Phi, \Gamma \rangle^* \quad (\star\star).$$

Adding $(\star)$ and $(\star\star)$, $\quad \langle \Gamma, \Phi \rangle = \langle \Phi, \Gamma \rangle^*$. Thus $\langle , \rangle$ becomes a B-valued inner product on X'' with aid of Lemma 3.5. Also, by Lemma 3.5, since norm on X'' gotten from this inner product coincides with the operator norm $\| \cdot \|_{X''}$, X'' is a Hilbert B-module with respect to the inner product we have introduced. □

THEOREM 3.7. *Under the same situation, there exists a module isomorphism of $(X'')'$ onto X'.*

Proof. For $F \in (X'')'$, define $\tau_F \in X'$ by $\tau_F(x) = F(\dot{x})$ $(x \in X)$ and for $\tau \in X'$, define $F_\tau \in (X'')'$ by $F_\tau(\Gamma) = \Gamma(\tau)^*$ $(\Gamma \in X'')$. i.e.,

$$\Psi_1 : (X'')' \longrightarrow X'(F \to \tau_F)$$

$$\Psi_2 : X' \longrightarrow (X'')'(\tau \rightarrow F_\tau)$$

Then

$$\begin{aligned}\Psi_1(F \cdot b)(x) &= \tau_{F \cdot b}(x) = (F \cdot b)(\dot{x}) \\ &= F(\dot{x})b = \tau_F(x) \cdot b = \Psi_1(F)(x) \cdot b.\end{aligned}$$

Also

$$\begin{aligned}\| \Psi_1(F) \| &= \| \tau_F \|_{X'} = sup\{\| \tau_F(x) \|_B : \| x \| \leq 1\} \\ &= sup\{\| F(\dot{x}) \|_B : \| \dot{x} \| \leq 1\} \\ &= \| F|_{\dot{X}} \| = \| F \|_{(X'')'} \ ([Lemma\ 3.4]).\end{aligned}$$

For Ψ_2, the same are true, and we have $\Phi_1\Big(\Psi_2(\tau)\Big)(x) = \Psi_2(\tau)(\dot{x}) = \dot{x}(\tau)^* = \tau(x)$,
$\Psi_2\Big(\Psi_1(F)\Big)(\dot{x}) = \dot{x}\Big(\Psi_1(F)\Big)^* = \Psi_1(F)(x) = F(\dot{x})$. Thus they are inverses of each other. □

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