Kangweon–Kyungki Math. Jour. **3** (1995), No. 1, pp. 107–114

ON WEAK–MIXING PROPERTY

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1. Introduction.

Let $(X, \mathcal{B}, m)$ be a probability space and let $T : X \to X$ a measure-preserving transformation. We introduce some results related ergodicity and mixing property of T. We try to extend 1-fold ergodicity and 1-fold mixing to r-fold ergodicity and r-fold mixing. We shall use the result, about sequences of real numbers, to obtain formulations of r-fold weak-mixing. We show that for a measure-preserving transformation T, r-fold weak-mixing properties of T are equivalent to r-fold strong-mixing properties of $T \times T$. Now, we shall introduce the concept of r-fold weak-mixing property and r-fold strong-mixing property. We show that basic properties of 1-fold mixing keep same properties in the case of r-fold mixing. We shall use that U is defined on functions by $Uf = f \circ T$, $\forall f \in L^p(X, \mathcal{B}, m)$, $p \geq 1$.

2. r–fold mixing.

DEFINITION 1. Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, m)$ and let $r \geq 1$.
(a) T is *r-fold ergodic* if $^\forall A_0, A_1, \cdots, A_r \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} m(A_0 \cap T^{-i}A_1 \cap \cdots \cap T^{-ri}A_r) = m(A_0)m(A_1)\cdots m(A_r).$$

(b) T is r-*fold weak-mixing* if $^\forall A_0, A_1, \cdots, A_r \in \mathcal{B}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} &|m(A_0 \cap T^{-i}A_1 \cap \cdots \cap T^{-ri}A_r) \\ &- m(A_0)m(A_1)\cdots m(A_r)| \\ &= 0.\end{aligned}$$

Received November 2, 1994.

(c) T is *r-fold strong-mixing* if $\forall A_0, A_1, \cdots, A_r \in \mathcal{B}$,

$$\lim_{n_1,\cdots,n_r\to\infty} m(A_0 \cap T^{-n_1}A_1 \cap \cdots \cap T^{-(n_1+\cdots+n_r)}A_r) = \prod_{j=0}^{r} m(A_j).$$

THEOREM 1. *Let $T : (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ be a measure- preserving transformation.*

(i) The following are equivalent :

(1) T is r-fold ergodic.

(2) For all $f_0, f_1, \cdots, f_r \in L^{r+1}(X, \mathcal{B}, m)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \int (f_0 U^i f_1 \cdots U^{ri} f_r) dm = \prod_{j=0}^{r} \int f_j dm.$$

(ii) The following are equivalent :

(1) T is r-fold weak-mixing.

(2) For all $f_0, f_1, \cdots, f_r \in L^{r+1}(X, \mathcal{B}, m)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \left| \int (f_0 U^i f_1 \cdots U^{ri} f_r) dm - \prod_{j=0}^{r} \int f_j dm \right| = 0.$$

(iii) The following are equivalent :

(1) T is r-fold strong-mixing.

(2) For all $f_0, f_1, \cdots, f_r \in L^{r+1}(X, \mathcal{B}, m)$,

$$\lim_{n_1,\cdots,n_r\to\infty} \int (f_0 U^{n_1} f_1 \cdots U^{n_1+\cdots+n_r} f_r)\, dm = \prod_{j=0}^{r} \int f_j dm.$$

Proof. (i),(ii) and (iii) are proved by using similar methods of [4].

REMARK. Every r-fold strong-mixing transformation implies r-fold weak-mixing and every r-fold weak-mixing transformation implies r-fold ergodic.

This is because if $\{a_n\}$ is a sequence of real numbers then

$$\lim_{n\to\infty} a_n = 0$$

implies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} |a_i| = 0$$

and this second condition implies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} a_i = 0.$$

We shall use the following result, about sequences of real numbers, to obtain other formulations of r-fold weak-mixing.

THEOREM 2. *If $\{a_n\}$ is a bounded sequence of real numbers then the following are equivalent :*

(i)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} |a_i| = 0.$$

(ii) There exists a subset J of $\mathbb{Z}^+$ of density zero

$$(i.e., \left(\frac{\text{cardinality}(J \cap \{0,1,\cdots,n-1\})}{n}\right) \to 0),$$

such that $\lim_{n\to\infty} a_n = 0$ provided $n \notin J$.

(iii)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} |a_i|^2 = 0.$$

Proof. See [7].

THEOREM 3. *If T is a measure-preserving transformation of a probability space $(X, \mathcal{B}, m)$, the followings are equivalent:*
(i) T is weak–mixing.
(ii) For all $A_0, A_1, \cdots, A_r$ of $\mathcal{B}$ there is a subset $J(A_0, A_1, \cdots, A_r)$ of $\mathbb{Z}^+$ of density zero such that

$$\lim_{\substack{n \notin J(A_0, A_1, \cdots, A_r) \\ n \to \infty}} m(A_0 \cap T^{-n} A_1 \cap \cdots \cap T^{-rn} A_r) = m(A_0) m(A_1) \cdots m(A_r).$$

(iii) For all $A_0, A_1, \cdots, A_r$ of $\mathcal{B}$ we have

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} &|m(A_0 \cap T^{-i} A_1 \cap \cdots \cap T^{-ri} A_r) \\ &- m(A_0) m(A_1) \cdots m(A_r)|^2 \\ &= 0. \end{aligned}$$

Proof. Apply Theorem 2 with

$$a_n = m(A_0 \cap T^{-n} A_1 \cap T^{-2n} A_2 \cap \cdots \cap T^{-rn} A_r) - m(A_0) m(A_1) \cdots m(A_r).$$

The following theorem gives a way of checking the mixing properties for examples by reducing the computations to a class of sets we can manipulate with.

THEOREM 4. *Let $(X, \mathcal{B}, m)$ be a probability space and let $\mathcal{T}$ be a semi-algebra that generates $\mathcal{B}$. Let $T : X \to X$ be a measure-preserving transformation. Then T is r-fold strang-mixing iff*
$^\forall C_0, C_1, \cdots, C_r \in \mathcal{T}$,

$$\begin{aligned} \lim_{n_1, \cdots, n_r \to \infty} &m(C_0 \cap T^{-n_1} C_1 \cap \cdots \cap T^{-(n_1 + \cdots + n_r)} C_r) \\ &= m(C_0) m(C_1) \cdots m(C_r). \end{aligned}$$

Proof. See [4].

3. r–fold weak–mixing.

We have main result. This result connects r-fold weak-mixing of T with r-fold ergodicity of $T \times T$. Also, this result relates to r-fold weak-mixing of T with r-fold weak-mixing of $T \times T$.

THEOREM 5. *If T is a measure-preserving transformation on a probability space $(X, \mathcal{B}, m)$ then the followings are equivalent:*
(i) T is r-fold weak-mixing .
(ii) $T \times T$ is r-fold ergodic .
(iii) $T \times T$ is r-fold weak-mixing .

Proof. ((i) $\Rightarrow$ (iii)). Let $A_0, A_1, \cdots, A_r \in \mathcal{B}$, $B_0, B_1, \cdots, B_r \in \mathcal{B}$. There exist subsets $J_1(A_0, A_1, \cdots, A_r)$, $J_2(B_0, B_1, \cdots, B_r)$ of $\mathbb{Z}^+$ of density zero such that

$$\lim_{\substack{n \notin J_1 \\ n \to \infty}} m(A_0 \cap T^{-n} A_1 \cap \cdots \cap T^{-rn} A_r) = m(A_0) m(A_1) \cdots m(A_r)$$

and

$$\lim_{\substack{n \notin J_2 \\ n \to \infty}} m(B_0 \cap T^{-n} B_1 \cap \cdots \cap T^{-rn} B_r) = m(B_0) m(B_1) \cdots m(B_r).$$

Then

$$\lim_{\substack{n \notin J_1 \cup J_2 \\ n \to \infty}} (m \times m)\{(A_0 \times B_0) \cap (T \times T)^{-n}(A_1 \times B_1) \cap \cdots \cap (T \times T)^{-rn}(A_r \times B_r)\}$$

$$= \lim_{\substack{n \notin J_1 \cup J_2 \\ n \to \infty}} m(A_0 \cap T^{-n} A_1 \cap \cdots \cap T^{-rn} A_r) \cdot m(B_0 \cap T^{-n} B_1 \cap \cdots \cap T^{-rn} B_r)$$

$$= m(A_0) m(A_1) \cdots m(A_r) m(B_0) m(B_1) \cdots m(B_r)$$

$$= (m \times m)(A_0 \times B_0)(m \times m)(A_1 \times B_1) \cdots (m \times m)(A_r \times B_r).$$

By Theorem 2 we know

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|(m\times m)[(A_0\times B_0)\cap(T\times T)^{-i}(A_1\times B_1)\cap\cdots$$
$$\cap(T\times T)^{-ri}(A_r\times B_r)]-\prod_{j=0}^{r}(m\times m)(A_j\times B_j)|$$
$$=0.$$

Since the measurable rectangles form a semi-algebra that generates $\mathcal{B}\times\mathcal{B}$, Theorem 4 asserts that $T\times T$ is r-fold weak-mixing.

((iii) $\Rightarrow$ (ii)). It is clear that (iii) implies (ii).

((ii) $\Rightarrow$ (i)). Let $A_0, A_1, \cdots, A_r \in \mathcal{B}$. We shall show

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\{m(A_0\cap T^{-i}A_1\cap\cdots\cap T^{-ri}A_r)-m(A_0)m(A_1)\cdots m(A_r)\}^2=0.$$

We have

$$\frac{1}{n}\sum_{i=1}^{n}m(A_0\cap T^{-i}A_1\cap\cdots\cap T^{-ri}A_r)$$
$$=\frac{1}{n}\sum_{i=1}^{n}(m\times m)\{(A_0\times X)\cap(T\times T)^{-i}(A_1\times X)\cap\cdots$$
$$\cap(T\times T)^{-ri}(A_r\times X)\}$$
$$\to(m\times m)(A_0\times X)(m\times m)(A_1\times X)\cdots(m\times m)(A_r\times X)$$
$$=m(A_0)m(A_1)\cdots m(A_r).$$

Also

$$\frac{1}{n}\sum_{i=1}^{n}\{m(A_0 \cap T^{-i}A_1 \cap \cdot\cdot \cap T^{-ri}A_r)\}^2$$
$$= \frac{1}{n}\sum_{i=1}^{n}(m \times m)\{(A_0 \times A_0) \cap (T \times T)^{-i}(A_1 \times A_1) \cap \cdots$$
$$\cap (T \times T)^{-ri}(A_r \times A_r)\}$$
$$\to (m \times m)(A_0 \times A_0)(m \times m)(A_1 \times A_1) \cdots$$
$$(m \times m)(A_r \times A_r)$$
$$= m(A_0)^2 m(A_1)^2 \cdots m(A_r)^2.$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n}\{m(A_0 \cap T^{-i}A_1 \cap \cdots \cap T^{-ri}A_r) - m(A_0)m(A_1)\cdots m(A_r)\}^2$$
$$= \frac{1}{n}\sum_{i=1}^{n}\{(m(A_0 \cap T^{-i}A_1 \cap \cdots \cap T^{-ri}A_r))^2$$
$$- 2m(A_0 \cap T^{-i}A_1 \cap \cdots \cap T^{-ri}A_r)m(A_0)m(A_1)\cdots m(A_r)$$
$$+ m(A_0)^2 m(A_1)^2 \cdots m(A_r)^2\}$$
$$\to 2m(A_0)^2 m(A_1)^2 \cdots m(A_r)^2 - 2m(A_0)^2 m(A_1)^2 \cdots m(A_r)^2$$
$$= 0.$$

Therefore T is r-fold weak-mixing by Theorem 3.

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