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A STUDY ON THE SUBMANIFOLDS OF A MANIFOLD GSX_n

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1. Introduction.

On a generalized Riemannian manifold X_n, we may impose a particular geometric structure by the basic tensor field $g_{\lambda\mu}$ by means of a particular connection $\Gamma_\lambda{}^\nu{}_\mu$. For example, Einstein's manifold X_n is based on the Einstein's connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by the Einstein's equations.

Many *recurrent* connections have been studied by many geometers, such as Datta and Singel([1]), M. Matsumoto, and E.M. Patterson.

The purpose of the present paper is to introduce the concept of the *g-recurrent connection* and to derive some generalized fundamental equations on the submanifolds of a *generalized semisymmetric g-recurrent manifold* GSX_n.

All considerations in this present paper deal with the general case $n \geq 2$ and all possible classes and indices of inertia.

2. Preliminaries.

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system y^ν, with coordinate transformation $y^\nu \to \bar{y}^\nu$, for which

$$(2.1) \qquad Det\left(\frac{\partial y}{\partial \bar{y}}\right) \neq 0.$$

The manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$, which may be split into a symmetric part $h_{\lambda\mu}$ and a skew-symmetric part $k_{\lambda\mu}$:

$$(2.2) \qquad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

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where

$$(2.3) \qquad \mathcal{G} = Det(g_{\lambda\mu}) \neq 0, \qquad \mathcal{H} = Det(h_{\lambda\mu}) \neq 0.$$

Hence, we may define a unique tensor $h^{\lambda\nu}$ by

$$(2.4) \qquad h_{\lambda\mu}h^{\lambda\nu} = \delta_\mu^\nu$$

and X_n is assumed to be connected by a real nonsymmetric connection $\Gamma_\lambda{}^\nu{}_\mu$ with the following transformation rule:

$$(2.5) \qquad \bar{\Gamma}_\lambda{}^\nu{}_\mu = \frac{\partial \bar{y}^\nu}{\partial y^\alpha}\left(\frac{\partial y^\beta}{\partial \bar{y}^\lambda}\frac{\partial y^\gamma}{\partial \bar{y}^\mu}\Gamma_\beta{}^\alpha{}_\gamma + \frac{\partial^2 y^\alpha}{\partial \bar{y}^\lambda \partial \bar{y}^\mu}\right).$$

This connection may also be decomposed into its symmetric part $\Lambda_\lambda{}^\nu{}_\mu$ and its skew-symmetric part $S_{\lambda\mu}{}^\nu$, called the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$:

$$\Gamma_\lambda{}^\nu{}_\mu = \Lambda_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu$$

where

$$(2.6) \qquad \Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}, \qquad S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda}{}^\nu{}_{\mu]}.$$

Now, we will define a manifold GSX_n.

It is well-known that a connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be semisymmetric if its torsion tensor is of the form

$$(2.7) \qquad S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda}X_{\mu]}$$

for an arbitrary vector $X_\mu \neq 0$.

A particular differential geometric structure may be imposed on X_n by the tensor field $g_{\lambda\mu}$ by means of the connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by the following g-recurrent condition:

$$(2.8) \qquad D_\omega g_{\lambda\mu} = -4X_\omega g_{\lambda\mu}.$$

Here, X_ω is a non-null vector and D_ω is the symbolic vector of the covariant derivative with respect to the connection $\Gamma_\lambda{}^\nu{}_\mu$.

DEFINITION 2.1. *The connection $\Gamma_\lambda{}^\nu{}_\mu$ which satisfies (2.8) is called a g-recurrent connection.*

DEFINITION 2.2. *A connection which is both semisymmetric and g-recurrent is called a GS connection.*

A generalized Riemannian manifold X_n on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through a GS connection is called an n-dimensional GS manifold and will be denoted by GSX_n.

The following theorem has been proved([6]).

THEOREM 2.3. *If the system (2.8) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$ in GSX_n, it must be of the form*

$$\Gamma_\lambda{}^\nu{}_\mu = \{{}_\lambda{}^\nu{}_\mu\} + 2\delta^\nu_\lambda X_\mu. \tag{2.9}$$

3. The induced connection on X_m of X_n $(m < n)$.

This section is a brief collection of basic concepts, results, and notations needed in the present paper. It is based on the results and notations of Chung et So ([4]).

AGREEMENT 3.1. *In our further considerations in the present paper, we use the following types of indices:*

(1) *Lower Greek indices α, β, γ,..., running from 1 to n and used for the holonomic components of tensors in X_n.*
(2) *Capital Latin indices A,B,C,..., running from 1 to n and used for the C-nonholonomic components of tensors in X_n at points of X_m.*
(3) *Lower Latin indices i, j, k,..., with the exception of x, y, and z, running from 1 to $m(< n)$.*
(4) *Lower Latin indices x, y, z, running from $m+1$ to n.*

The summation convention is operative with respect to each set of the above indices within their range, with exception of x, y, z.

Let X_m be a submanifold of X_n defined by a system of sufficiently differentiable equations

$$y^\nu = y^\nu(x^1, \ldots\ldots, x^m) \tag{3.1}$$

where the matrix of derivatives

$$B_i^\nu = \frac{\partial y^\nu}{\partial x^i}$$

is of rank m. Hence at each point of X_m, there exists *the first set* $\{B_i^\nu, \underset{x}{N}^\nu\}$ *of* n *linearly independent nonnull vectors.*

The m vectors B_i^ν are tangential to X_m and the $n-m$ vectors $\underset{x}{N}^\nu$ are normal to X_m and mutually orthogonal. That is

$$h_{\alpha\beta} B_i^\alpha \underset{x}{N}^\beta = 0, \qquad h_{\alpha\beta} \underset{x}{N}^\alpha \underset{y}{N}^\beta = 0 \qquad \text{for } x \neq y. \tag{3.2}$$

The process of determining the set $\{\underset{x}{N}^\nu\}$ is not unique unless $m = n-1$.

However, we may choose their magnitudes such that

$$h_{\alpha\beta} \underset{x}{N}^\alpha \underset{x}{N}^\beta = \varepsilon_x \tag{3.3}$$

where $\varepsilon_x = \pm 1$ according as the left-hand side of (3.3) is positive or negative.

Put

$$E_A^\nu = \begin{cases} B_i^\nu, & \text{if } A = 1, \ldots, m(=i) \\ \underset{x}{N}^\nu, & \text{if } A = m+1, \ldots, n(=x). \end{cases} \tag{3.4}$$

Since $\{E_A^\nu\}$ is a set of n linearly independent vectors in X_n at points of X_m, there exists a unique second set $\{E_\lambda^A\}$ of n linearly independent vectors at points of X_m such that

$$E_\lambda^A E_A^\nu = \delta_\lambda^\nu, \qquad E_\alpha^A E_B^\alpha = \delta_B^A. \tag{3.5}$$

Put

$$E_\lambda^A = \begin{cases} B_\lambda^i, & \text{if } A = 1, \ldots, m(=i) \\ \overset{x}{N}_\lambda, & \text{if } A = m+1, \ldots, n(=x), \end{cases} \tag{3.6}$$

$$B_\lambda^\nu = B_\lambda^i B_i^\nu. \tag{3.7}$$

Then, we can see that the following relations hold in virtue of (3.5):

$$B^i_\alpha B^\alpha_j = \delta^i_j, \quad \overset{x}{N}_\alpha \underset{y}{N}^\alpha = \delta^x_y, \quad B^i_\alpha \underset{x}{N}^\alpha = \overset{x}{N}_\alpha B^\alpha_i = 0, \tag{3.8}$$

$$B^\nu_\lambda = \delta^\nu_\lambda - \sum_x \overset{x}{N}_\lambda \underset{x}{N}^\nu, \quad B^\alpha_\lambda \overset{x}{N}_\alpha = B^\nu_\alpha \underset{x}{N}^\alpha = 0. \tag{3.9}$$

In virtue of (3.8), we note that the vectors B^i_λ form the second set of linearly independent vectors tangential to X_m. We also note that the set $\{\overset{x}{N}_\lambda\}$ is the *second set* of $n-m$ nonnull vectors normal to X_m, which are linearly independent and mutually orthogonal. Now, we are ready to introduce the following concepts of *C-nonholonomic frame of reference* and *induced tensors*.

DEFINITION 3.2. *The sets $\{E^\nu_A\}$ and $\{E^A_\lambda\}$ is referred to as the C-nonholonomic frame of reference in X_n at points of X_m. This frame gives rise to C-nonholonomic components of tensors in X_n.*

If $T^{\nu\cdots\cdots}_{\lambda\cdots\cdots}$ are holonomic components of a tensor in X_n, then at points of X_m its C-nonholonomic components $T^{A\cdots\cdots}_{B\cdots\cdots}$ are defined by

$$T^{A\cdots\cdots}_{B\cdots\cdots} = T^{\alpha\cdots\cdots}_{\beta\cdots\cdots} E^A_\alpha \cdots\cdots E^\beta_B \cdots\cdots \tag{3.10}$$

In particular, the quantities

$$T^{i\cdots\cdots}_{j\cdots\cdots} = T^{\alpha\cdots\cdots}_{\beta\cdots\cdots} B^i_\alpha \cdots\cdots B^\beta_j \cdots\cdots \tag{3.11}$$

are components of a tensor in X_m and are called the components of the induced tensor of $T^{\nu\cdots\cdots}_{\lambda\cdots\cdots}$ on X_m of X_n.

Therefore, the induced metric tensor g_{ij} on X_m of $g_{\lambda\mu}$ in X_n may be given by

$$g_{ij} = g_{\alpha\beta} B^\alpha_i B^\beta_j. \tag{3.12}$$

In virtue of (3.5), we know that

$$T^{\nu\cdots\cdots}_{\lambda\cdots\cdots} = T^{A\cdots\cdots}_{B\cdots\cdots} E^\nu_A \cdots\cdots E^B_\lambda \cdots\cdots \tag{3.13}$$

As a consequence of (3.13), we have

$$\begin{aligned} h_{\lambda\mu} &= h_{ij} B^i_\lambda B^j_\mu + \sum_x \varepsilon_x \overset{x}{N}_\lambda \overset{x}{N}_\mu \\ h^{\lambda\nu} &= h^{ij} B^\lambda_i B^\nu_j + \sum_x \varepsilon_x \underset{x}{N}^\lambda \underset{x}{N}^\nu. \end{aligned} \tag{3.14}$$

As another consequence of (3.13), we have

THEOREM 3.3. *At each point of X_m any vector X_λ in X_n may be expressed as the sum of two vectors $X_i B^i_\lambda$ and $\sum_x X_x \overset{x}{N}_\lambda$, the former tangential to X_m and the latter normal to X_m. That is*

$$X_\lambda = X_i B^i_\lambda + \sum_x X_x \overset{x}{N}_\lambda \tag{3.15a}$$

or equivalently,

$$X^\nu = X^i B^\nu_i + \sum_x X^x \underset{x}{N}^\nu \tag{3.15b}$$

where

$$X_i = X_\alpha B^\alpha_i, \qquad X_x = X_\alpha \underset{x}{N}^\alpha, \qquad X_x = \varepsilon_x X^x$$

$$X^i = X^\alpha B^i_\alpha, \qquad X^x = X^\alpha \overset{x}{N}_\alpha.$$

Furthermore, $X_i(X^i)$ are components of a tangent vector relative to the transformations of X_m, while $X_x(X^x)$ is invariant relative to the transformations of X_m and X_n.

4. The induced connection on X_m of GSX_n $(m < n)$.

DEFINITION 4.1. *If $\Gamma_\lambda{}^\nu{}_\mu$ is a connection on X_n, the connection $\Gamma^k_i{}_j$ defined by*

$$\Gamma_i{}^k{}_j = B^k_\gamma (B^\gamma_{ij} + \Gamma_\alpha{}^\gamma{}_\beta B^\alpha_i B^\beta_j), \qquad B^\gamma_{ij} = \frac{\partial B^\gamma_i}{\partial x^j} = \frac{\partial^2 y^\gamma}{\partial x^i \partial x^j} \tag{4.1}$$

is called the induced connection of $\Gamma_\lambda{}^\nu{}_\mu$ on X_m of X_n.

The following statements have been already proved([3]):

(a) The torsion tensor $S_{ij}{}^k$ of the induced connection $\Gamma_i{}^k{}_j$ is the induced tensor of the torsion tensor $S_{\lambda\mu}{}^\nu$ of the connection $\Gamma_\lambda{}^\nu{}_\mu$. That is

$$S_{ij}{}^k = S_{\alpha\beta}{}^\gamma B^\alpha_i B^\beta_j B^k_\gamma. \tag{4.2}$$

(b) The induced connection $\{{}_i{}^k{}_j\}$ of $\{{}_\lambda{}^\nu{}_\mu\}$ is the Christoffel symbol defined by h_{ij}. That is

$$\{{}_i{}^k{}_j\} = \frac{1}{2} h^{kp} (\partial_i h_{jp} + \partial_j h_{ip} - \partial_p h_{ij}). \tag{4.3}$$

THEOREM 4.2. *On an X_m of GSX_n, the induced connection $\Gamma_i{}^k{}_j$ is of the form*

$$\Gamma_i{}^k{}_j = \{{}_i{}^k{}_j\} + 2\delta_i^k X_j. \tag{4.4}$$

Here $\{{}_i{}^k{}_j\}$ are the induced Christoffel symbols defined by (4.3) and X_j is the induced vector on X_m of a vector $X_\mu \neq 0$ determining $\Gamma_\lambda{}^\nu{}_\mu$. That is

$$X_i = X_\alpha B_i^\alpha. \tag{4.5}$$

Proof. In virtue of (4.1), (4.3), (2.10), and (3.5), we have (4.4).

Let $\overset{o}{D}_j$ be the symbolic vector of the generalized covariant derivative with respect to the $x's$. That is

$$\overset{o}{D}_j B_i^\alpha = B_{ij}^\alpha + \Gamma_\beta{}^\alpha{}_\gamma B_i^\beta B_j^\gamma - \Gamma_i{}^k{}_j B_k^\alpha. \tag{4.6}$$

Then the vector $\overset{o}{D}_j B_i^\alpha$ in X_n is normal to X_m and is given by Chung et al ([3]).

$$\overset{o}{D}_j B_i^\alpha = -\sum_x \overset{x}{\Omega}_{ij} \underset{x}{N}^\alpha \tag{4.7}$$

where

$$\overset{x}{\Omega}_{ij} = -(\overset{o}{D}_j B_i^\alpha) \overset{x}{N}_\alpha. \tag{4.8}$$

And we know that the tensors $\overset{x}{\Omega}_{ij}$ are the induced tensors on X_m of the tensor $D_\beta \overset{x}{N}_\alpha$ in X_n. That is

$$\overset{x}{\Omega}_{ij} = (D_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta. \tag{4.9}$$

The tensor $\overset{x}{\Omega}_{ij}$ will be called the generalized coefficients of the second fundamental form of X_m.

THEOREM 4.3. *The coefficients* $\overset{x}{\Omega}_{ij}$ *of the submanifold* X_m *of* GSX_n *are given by*

$$\overset{x}{\Omega}_{ij} = (\nabla_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta \tag{4.10}$$

where ∇_β *denotes the symbolic vector of the covariant derivative with respect to* $\{\lambda^\nu{}_\mu\}$.

Proof. In virtue of (2.10), (4.9),and (3.8), the relation (4.10) follows:

$$\begin{aligned}
\overset{x}{\Omega}_{ij} &= (D_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta \\
&= (\partial_\beta \overset{x}{N}_\alpha - \Gamma_\alpha{}^\gamma{}_\beta \overset{x}{N}_\gamma) B_i^\alpha B_j^\beta \\
&= [\partial_\beta \overset{x}{N}_\alpha - (\{{}_\alpha{}^\gamma{}_\beta\} + 2\delta_\alpha^\gamma X_\beta) \overset{x}{N}_\gamma] B_i^\alpha B_j^\beta \\
&= (\partial_\beta \overset{x}{N}_\alpha - \{{}_\alpha{}^\gamma{}_\beta\} \partial_\beta \overset{x}{N}_\gamma) B_i^\alpha B_j^\beta - 2X_\beta \overset{x}{N}_\alpha B_i^\alpha B_j^\beta \\
&= (\nabla_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta.
\end{aligned}$$

REMARK 4.4. *The following identity*

$$\overset{o}{D}_j B_i^\alpha = -\sum_x \overset{x}{\Lambda}_{ij} \underset{x}{N}^\alpha \qquad \text{where} \qquad \overset{x}{\Lambda}_{ij} = (\nabla_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta \tag{4.11}$$

(Generalized Gauss formulas for an X_m *of* GSX_n*)*

is a direct consequence of (4.10).

In our subsequent considerations, we frequently use the following C-nonholonomic components:

$$k_{ix} = -k_{xi} = k_{\alpha\beta} B_i^\alpha \underset{x}{N}^\beta = g_{\alpha\beta} B_i^\alpha \underset{x}{N}^\beta. \tag{4.12}$$

THEOREM 4.5. *On an* X_m *of* GSX_n*, the induced tensor of* $D_\omega g_{\lambda\mu}$ *may be given by*

$$D_\omega g_{\lambda\mu} B_i^\lambda B_j^\mu B_k^\mu = D_k g_{ij} + 2\sum_x k_{x[j} \overset{x}{\Lambda}_{i]k}, \tag{4.13}$$

where D_k is the symbolic vector of the covariant derivative with respect to Γ_{ij}^k.

Proof. In virtue of (3.12), (3.9), (4.11), it follows from (3.11) that

$$\begin{aligned} D_k g_{ij} &= \overset{o}{D}_k g_{ij} \\ &= \overset{o}{D}_k (g_{\lambda\mu} B_i^\lambda B_j^\mu) \\ &= (\overset{o}{D}_k g_{\lambda\mu}) B_i^\lambda B_j^\mu + g_{\lambda\mu}[(\overset{o}{D}_k B_i^\lambda) B_j^\mu + B_i^\lambda (\overset{o}{D}_k B_j^\mu)] \\ &= (D_\omega g_{\lambda\mu}) B_i^\lambda B_j^\mu B_k^\omega - g_{\lambda\mu}(\sum_x \overset{x}{A}_{ik} \underset{x}{N}^\lambda B_j^\mu + \sum_x \overset{x}{A}_{jk} \underset{x}{N}^\mu B_i^\lambda) \\ &= (D_\omega g_{\lambda\mu}) B_i^\lambda B_j^\mu B_k^\omega - k_{\lambda\mu} \sum_x (-\overset{x}{A}_{ik} B_j^\lambda \underset{x}{N}^\mu + \overset{x}{A}_{jk} B_i^\lambda \underset{x}{N}^\mu) \\ &= (D_\omega g_{\lambda\mu}) B_i^\lambda B_j^\mu B_k^\omega - \sum_x (-\overset{x}{A}_{ik} k_{jx} + \overset{x}{A}_{jk} k_{ix}) \\ &= (D_\omega g_{\lambda\mu}) B_i^\lambda B_j^\mu B_k^\omega - 2 \sum_x k_{x[j} \overset{x}{A}_{i]k}. \end{aligned}$$

The following theorem is an immediate consequence of (4.13).

THEOREM 4.6. *On an X_m of GSX_n, a necessary and sufficient condition for the induced connection $\Gamma_i{}^k{}_j$ to be g-recurrent is*

$$\sum_x k_{x[i} \overset{x}{A}_{j]k} = 0.$$

Now we are going to derive the generalized *Weingarten equations* for an X_m of GSX_n.

Let

$$\underset{jx}{M}^\alpha = \overset{o}{D}_j \underset{x}{N}^\alpha. \tag{4.14}$$

Then the relations (3.15) give

$$\underset{jx}{M}^\alpha = \underset{jx}{M}^i B_i^\alpha + \sum_y \underset{jx}{M}^y \underset{y}{N}^\alpha \tag{4.15}$$

where

$$(4.16)\qquad \underset{jx}{M}{}^i = \underset{jx}{M}{}^\alpha B^i_\alpha = (D_\gamma \underset{x}{N}{}^\alpha) B^i_\alpha B^\gamma_j$$
$$\underset{jx}{M}{}^y = \underset{jx}{M}{}^\alpha \overset{y}{N}_\alpha = (D_\gamma \underset{x}{N}{}^\alpha) \overset{y}{N}_\alpha B^\gamma_j.$$

THEOREM 4.7. *On an X_m of GSX_n, the induced vector $\underset{jx}{M}{}^i$ of $\underset{jx}{M}{}^\alpha$ is given by*

$$(4.17)\qquad \underset{jx}{M}{}^i = \varepsilon_x h^{im} \overset{x}{\Lambda}_{mj}.$$

Proof. In virtue of (2.14), (3.2), (3.3), (4.11) and (4.16), we have

$$\begin{aligned}
\underset{jx}{M}{}^i &= (\partial_\gamma \underset{x}{N}{}^\beta + \Gamma^\beta_{\epsilon\gamma} \underset{x}{N}{}^\epsilon) B^i_\beta B^\gamma_j \\
&= (\nabla_\gamma \underset{x}{N}{}^\beta) B^i_\beta B^\gamma_j \\
&= \varepsilon_x h^{im} (\nabla_\gamma \overset{x}{N}_\epsilon) B^\epsilon_m B^\gamma_j \\
&= \varepsilon_x h^{im} \overset{x}{\Lambda}_{mj}.
\end{aligned}$$

THEOREM 4.8. *On an X_m of GSX_n, the C-nonholonomic components $\underset{jx}{M}{}^y$ of $\underset{jx}{M}{}^\alpha$ are given by*

$$(4.18)\qquad \underset{jx}{M}{}^y = \varepsilon_y \underset{x}{\overset{y}{H}}{}_\gamma B^\gamma_j + 2\delta^y_x X_j \quad \text{where} \quad \underset{x}{\overset{y}{H}}{}_\gamma = \varepsilon_y (\nabla_\gamma \underset{x}{N}{}^\alpha) \overset{y}{N}_\alpha.$$

Proof. In virtue of (2.10), (3.8), (4.16), we can obtain (4.18).

$$\begin{aligned}
\underset{jx}{M}{}^y &= (D_\gamma \underset{x}{N}{}^\beta) \overset{y}{N}_\beta B^\gamma_j \\
&= [\partial_\gamma \underset{x}{N}{}^\beta + (\{{}_\alpha{}^\beta{}_\gamma\} + 2\delta^\beta_\alpha X_\gamma) \underset{x}{N}{}^\alpha] \overset{y}{N}_\beta B^\gamma_j \\
&= (\nabla_\gamma \underset{x}{N}{}^\beta) \overset{y}{N}_\beta B^\gamma_j + 2X_\gamma \underset{x}{N}{}^\beta \overset{y}{N}_\beta B^\gamma_j \\
&= (\nabla_\gamma \underset{x}{N}{}^\beta) \overset{y}{N}_\beta B^\gamma_j + 2X_\gamma \delta^y_x B^\gamma_j \\
&= \varepsilon_y \underset{x}{\overset{y}{H}}{}_\gamma B^\gamma_j + 2\delta^y_x X_j.
\end{aligned}$$

THEOREM 4.9. *On an X_m of GSX_n, we have generalized Weingarten equations on an X_m of GSX_n:*

$$\overset{o}{D}_j \underset{x}{N}^\alpha = (\varepsilon_x h^{im} \overset{x}{A}_{mj}) B_i^\alpha + \sum_y (\varepsilon_y \underset{x}{\overset{y}{H}}_\gamma B_j^\gamma + 2\delta_x^y X_j) \underset{y}{N}^\alpha. \tag{4.19}$$

Proof. Substituting (4.17), (4.18) into (4.15), we have (4.19).

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