Kangweon-Kyungki Math. Jour. **3** (1995), No. 1, pp. 131–133

# FRACTIONAL IDEALS WITH RESPECT TO FINITE ALGEBRAIC EXTENSIONS

SANG–KYU BAHK*, GI–IK KIM AND HEUNG–JOON OH

Throughout this paper all rings will be a commutative integral domain with identity. Let $D$ be an integral domain with quotient field $K$ and let $\bar{D}$ be the integral closure in the quotient field of $D$. The group of divisibility of $D$ is the group $G(D) = K^*/U(D)$ where $K^*$ is the multiplicative group of $K$ and $U(D)$ is the group of units of $D$. $G(D)$ is partially ordered by $aU(D) \leq bU(D)$ if and only if $a \mid b$ in $D$. Several classes of integral domains $D$ are completely characterized by the order properties of $G(D)$; for example, $D$ is a valuation domain if and only if $G(D)$ is totally ordered. A fractional ideal $I$ of $D$ is a $D$-submodule of $K$ for which there is a non-zero element $x$ of $D$ with $xI \subset D$. The set $F(D)$ of non-zero fractional ideals of $D$ forms a commutative monoid under multiplication. A valuation domain $V$ is said to be discrete if every $P$-primary ideal of $V$ is a power of $P$. Note that we are not requiring a discrete valuation domain to have rank one. An $n$-dimensional valuation domain $V$ is discrete if and only if $G(V) \approx \mathbf{Z}^n$(as abelian groups) and in this case $G(V)$ is actually order-isomorphic to the lexicographic direct sum of $n$ copies of $\mathbf{Z}$. For notation and terminology not defined here, the reader is referred to [3] and [5].

In this paper, we show that if $F(D)$ is finitely generated, and $L$ is a finite algebraic extension field of $K$, then $F(D')$ is finitely generated where $D'$ is the integral closure of $D$ in $L$.

We collect in Theorem 0 results from [1] and [2] which will be used throughout this paper without further reference.

THEOREM 0. *Let $\bar{D}$ be the integral closure of $D$ in the quotient field $K$ of $D$.*

(1) *If $F(D)$ is finitely generated, then $G(D)$ is finitely generated.*

---

Received October 13, 1994.

* This work supported by Inha University Research Foundation 1994–1995.

(2) *$G(D)$ is finitely generated if and only if $G(\bar{D})$ is finitely generated and $\bar{D}/[D:\bar{D}]$ is finite, where $[D:\bar{D}] = \{d \in D \mid d\bar{D} \subseteq D\}$ is the conductor.*
(3) *$F(D)$ is finitely generated if and only if $\bar{D}$ is a semiquasilocal Bezout domain such that $\bar{D}_M$ is a finite rank discrete valuation domain for each maximal ideal $M$ of $\bar{D}$ and $\bar{D}/[D:\bar{D}]$ is finite.*
(4) *$F(D)$ is finitely generated if and only if $F(\bar{D})$ is finitely generated and $\bar{D}/[D:\bar{D}]$ is finite.*

*Proof.* (1) [2, Theorem 2.3]. (2) [1, Theorem 3]. (3),(4) [2, Theorem 5.3].

THEOREM 1. *Let $D$ be an integral domain with $F(D)$ finitely generated. Let $D'$ be the integral closure of $D$ in a finite algebraic extension field $L$ of the quotient field $K$ of $D$. Then $F(D')$ is finitely generated.*

*Proof.* Suppose $F(D)$ is finitely generated. Then $F(\bar{D})$ is finitely generated. Since $D' = \bar{D}'$, we may assume $D$ is integrally closed. Since $G(D)$ is finitely generated, $G(D')$ is also finitely generated by [4, Theorem 3.13]. Let $M$ be a maximal ideal of $D'$ and let $M \cap D = P$. Then $D_P$ is a finite rank discrete valuation domain, rank $G(D_P) =$ rank $G(D'_M)$, and $G(D'_M)$ is free. Since dim $D$ = dim $D'$, we have dim $D'$ = rank $G(D'_M)$. Hence $D'_M$ is a finite rank discrete valuation domain and so $F(D')$ is finitely generated.

If $D$ is Noetherian, then $\bar{D}$ is a Krull domain and so $\bar{D} = \bigcap D_P$, the intersection being taken over all height-one prime ideals of $\bar{D}$. Then $G(\bar{D})$ can be canonically embedded in the direct sum of the groups $G(D_P)$. Thus $G(\bar{D})$ is free, so $0 \longrightarrow U(\bar{D})/U(D) \longrightarrow G(D) \longrightarrow G(\bar{D}) \longrightarrow 0$ splits. Hence $G(D) \approx G(\bar{D}) \oplus U(\bar{D})/U(D) \approx \mathbf{Z}^{\alpha} \oplus U(\bar{D})/U(D)$, where $\alpha =$ rank $G(\bar{D}) \leq |X^{(1)}(\bar{D})|$ and $X^{(1)}(\bar{D})$ is the set of height-one prime ideals of $\bar{D}$. If $F(D)$ is finitely generated, then $G(D)$ and $G(\bar{D})$ are finitely generated and so $|X^{(1)}(\bar{D})| < \infty$. Thus $\bar{D}$ is a semilocal PID. Hence $D$ is a one-dimensional semilocal domain, rank $G(D)$ = rank $G(\bar{D})$ = $|X^{(1)}(\bar{D})| \geq |X^{(1)}(D)|$.

THEOREM 2. *Let $D$ be a one-dimensional semilocal domain such that the residue field of $D$ with respect to each maximal ideal is finite. If $F(D')$ is finitely generated and $D'$ is a finite $D$-module, then $F(D)$ is finitely generated.*

*Proof.* Note that if $D$ is Noetherian, then $F(D)$ is finitely generated if and only if $\bar{D}$ is a semilocal PID and $\bar{D}/[D:\bar{D}]$ is finite if and only if $G(D)$ is finitely generated. By [4, Theorem 3.13], $G(D)$ is finitely generated and hence $F(D)$ is finitely generated.

## References


[1] D. D. Anderson, *Integral domains with finitely generated groups of divisibility*, Pro. of the AMS. **112** (1991), 613-618.

[2] D. D. Anderson, J. L. Mott and J. Park, *Finitely generated monoids of fractional ideals*, Comm. in Algebra **21** (1993), 615-634.

[3] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, New York, 1972.

[4] B. Glastad and J. L. Mott, *Finitely generated groups of divisibility*, contemporary Math. **8** (1982), 231-247.

[5] I. Kaplansky, *Commutative rings, rev. ed.*, Univ. of Chicago Press, Chicago, 1974.



Sang-Kyu Bahk
Department of Mathematics
Inha University
Incheon, 402–751, Korea

Gi-Ik Kim and Heung-Joon Oh
Department of Mathematics
Myongji University
Yongin, 449–728, Korea