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FUZZY TOPOLOGY AND ITS APPLICATIONS

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1. Introduction.

The concept of a fuzzy set, introduced by Zadeh[6], was applied by Chang[2] in generalizing some of the basic concepts of general topology. Liu[3, 4] used the concept 'L-quasi-coincident' in fuzzy set theory and fuzzy topology to redefine L-fuzzy proximity space in such a way that it takes fuzzy proximity space as a special case, and proved some results analogous to those that hold for ordinary proximity spaces.

In the present paper, we provide a definition of L-fuzzy proximity-base in order to prove that the set of all L-fuzzy proximities on a set X forms a complete lattice. Let $f : X \to (Y, \delta_2)$ be a function. We show that there exists the coarsest L-fuzzy proximity on X such that f is a L- proximity map.

In this paper, $L = < L, \leq, \wedge, \vee, ' >$ denotes a completely distributive lattice with order-reversing involution. Let 0 be the least element and 1 the greatest element in L. Suppose X is a nonempty set. An *L-fuzzy set* in X is a map $A : X \to L$, and L^X will denote the family of all L-fuzzy sets in X. It is clear that $L^X = < L^X, \leq, \wedge, \vee, ' >$ is a completely distributive lattice with order reversing involution, which has the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$, where $\mathbf{0}(x) = 0$, $\mathbf{1}(x) = 1$ for any $x \in X$.

Let δ be a binary relation on L^X, i.e. $\delta \subset L^X \times L^X$. $(A, C) \in \delta$ and $(A, C) \notin \delta$ are denoted by $A\delta C$ and $A \not\delta C$, respectively. A binary relation δ on L^X is called a *L-fuzzy proximity* on X iff it satisfies the following axioms: For any $A, C, D, E \in L^X$

(LFP1) $A\delta C$ implies $C\delta A$.

(LFP2) If A and C are L-quasi-coincident (i.e. $A \not\leq C'$), then $A\delta C$.

(LFP3) If $A\delta C$, $A \leq D$ and $C \leq E$, then $D\delta E$.

(LFP4) $\mathbf{0} \not\delta \mathbf{1}$.

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(LFP5) If $A \not\delta D$ and $C \not\delta D$, then $(A \vee C) \not\delta D$.

(LFP6) $A \not\delta C$ implies that there exists $D \in L^X$ such that $A \not\delta D$ and $D' \not\delta C$.

The pair (X, δ) is called an *L-fuzzy proximity space.*

Let (X, δ) be an L-fuzzy proximity space. For any $A \in L^X$, We define $i : L^X \to L^X$ by

$$i(A) = \bigvee \{C \in L^X \mid C \not\delta A'\}.$$

Then i is an L-fuzzy interior operator on X, and thus

$$\mathcal{F}(\delta) = \{A \in L^X \mid A = i(A)\}$$

is an L-fuzzy topology on X. It is called the L-fuzzy topology induced by δ.[3]

Let f be a function from a set X into a set Y. If $D \in L^Y$, then the *inverse* of D corresponding to f, written as $f^{-1}[D]$, is a L-fuzzy set in X which is defined by $f^{-1}[D](x) = D(f(x))$ for all $x \in X$. Conversely, let A be a L-fuzzy set in X. The *image* of A by f, written as $f[A]$, is a L-fuzzy set in Y which is given by

$$f[A](y) = \begin{cases} 0, & \text{if } y \notin f(X) \\ \bigvee \{A(x) \mid f(x) = y\}, & \text{otherwise} \end{cases}$$

for all y in Y. Clearly, f^{-1} preserves complementation , arbitrary unions and arbitrary intersections and $f[\bigvee_i A_i] = \bigvee_i f[A_i]$. Also we have easily $f[f^{-1}[D]] \leq D$ and $f^{-1}[f[D]] \geq A$.[1]

Let δ_1 and δ_2 be two binary relations on L^X. We say that δ_2 is *finer than* δ_1 or that δ_1 is *coarser than* δ_2, denoted by $\delta_1 \preceq \delta_2$ iff for any A, $C \in L^X$, $A\delta_2 C$ implies $A\delta_1 C$. Let $\Delta(X)$ denote the set of all L-fuzzy proximities on X and $\delta_1, \delta_2, \delta_3 \in \Delta(X)$. Then:

(1) $\delta_1 \preceq \delta_1$.

(2) $\delta_1 \preceq \delta_2$, $\delta_2 \preceq \delta_1$ implies $\delta_1 = \delta_2$.

(3) $\delta_1 \preceq \delta_2$, $\delta_2 \preceq \delta_3$ implies $\delta_1 \preceq \delta_3$.

That is to say, $\Delta(X)$ is a partially ordered set with the binary relation $\preceq$.

Let A, $C \in L^X$. Define L-fuzzy proximities δ_1 and δ_2 on X as follows:

$$A\delta_1 C \quad \text{iff} \quad A \neq \mathbf{0} \quad \text{and} \quad C \neq \mathbf{0},$$

$$A\delta_2 C \quad \text{iff} \quad A \not\leq C'.$$

Then it is obvious that δ_1 is the coarsest L-fuzzy proximity in $\Delta(X)$, and δ_2 is the finest L-fuzzy proximity in $\Delta(X)$, so that δ_1 induces the L-fuzzy indiscrete topology, $\mathcal{F}(\delta_1) = \{ \mathbf{0}, \mathbf{1} \}$, and δ_2 induces the L-fuzzy discrete topology, $\mathcal{F}(\delta_2) = L^X$.

2. L-Fuzzy Proximity Base.

Now, we generalize the notion of a proximity-base in [5] to L-fuzzy sets. So we provide the following definition of an L-fuzzy proximity-base in order to prove that the set of all L-fuzzy proximities on a set X forms a complete lattice.

DEFINITION 1. A binary relation $\mathcal{B}$ on L^X is called an *L-fuzzy proximity base* on a nonempty set X iff it satisfies the following axioms :

(LFB1) $A\mathcal{B}C$ implies $C\mathcal{B}A$.
(LFB2) If $A \not\leq C'$, then $A\mathcal{B}C$.
(LFB3) If $A\mathcal{B}C$, $A \leq D$ and $C \leq E$, then $D\mathcal{B}E$.
(LFB4) $\mathbf{0}\not\mathcal{B}\,\mathbf{1}$.
(LFB5) $A\not\mathcal{B}C$ implies that there exists $D \in L^X$ such that $A\not\mathcal{B}D$ and $D'\not\mathcal{B}C$.

Let $J_m = \{1, 2, \cdots, m\}$. Let $\mathcal{B}$ be an L-fuzzy proximity base on a set X and let a binary relation $\delta^* = \delta(\mathcal{B})$ on L^X be defined as follows: $A\delta^*C$ iff given any two families $\{A_i \mid i \in J_m\}$ and $\{C_j \mid j \in J_n\}$ with $A = \bigvee_{i\in J_m} A_i$ and $C = \bigvee_{j\in J_n} C_j$, where $A_i, C_j \in L^X$, there exists a pair $(i,j) \in J_m \times J_n$ such that $A_i\mathcal{B}C_j$.

LEMMA 1. *the binary relation δ^*, which is defined above, is the coarsest L-fuzzy proximity on X, which is finer than the L-fuzzy proximity base $\mathcal{B}$. Here we say that the fuzzy proximity $\delta^* = \delta(\mathcal{B})$ is generated by the L-fuzzy proximity base $\mathcal{B}$.*

Proof. Let $A\delta^*C$. Then given any two families $\{A_i \mid i \in J_m\}$ and $\{C_j \mid j \in J_n\}$ with $A = \bigvee_i A_i$ and $C = \bigvee_j C_j$, there exists a pair $(i,j) \in J_m \times J_n$ such that $A_i\mathcal{B}C_j$. Since $A_i \leq A$ and $C_j \leq C$, $A\mathcal{B}C$ because of (LFB3). Hence we have $\mathcal{B} \preceq \delta^*$. Firstly, we show that δ^* is an L-fuzzy proximity on X.

(LFP1) It is satisfied because of (LFB1) and the definition of δ^*.

(LFP2) Let $A \not\leq C'$ and $A = \bigvee_i A_i$, $C = \bigvee_j C_j$. Since $A \not\leq (\bigvee_j C_j)' = \bigwedge_j C_j'$, there exist $x \in X$ such that $A(x) \not\leq \bigwedge_j C_j'(x)$. Hence there exist $j_0 \in J_n$ such that $A(x) \not\leq C_{j_0}'(x)$. And similarly, since $\bigvee_i A_i = A$, there exists $i_0 \in J_m$ such that $A_{i_0} \not\leq C_{j_0}'$. So $A_{i_0} \mathcal{B} C_{j_0}$. Hence $A\delta^*C$.

(LFP3) Let $A\delta^*C$, $A \leq D$ and $C \leq E$. Given any families $\{A_i \mid i \in J_m\}$, $\{C_j \mid j \in J_n\}$, $\{D_p \mid p \in J_k\}$ and $\{E_q \mid q \in J_l\}$ with $A = \bigvee_i A_i$, $C = \bigvee_j C_j$, $D = \bigvee_p D_p$ and $E = \bigvee_q E_q$. Since $A\delta^*C$, there exists a pair $(i,j) \in J_m \times J_n$ such that $A_i \mathcal{B} C_j$. Since $A_i \leq A \leq D = \bigvee_p D_p$, there exist $p_0 \in J_k$ such that $A_i \leq D_{p_0}$. Also, since $C_j \leq C \leq E = \bigvee_q E_q$, there exist $q_0 \in J_l$ such that $C_j \leq E_{q_0}$. Hence, by (LFB3), $D_{p_0} \mathcal{B} E_{q_0}$. Thus $D\delta^*E$.

(LFP4) Suppose $\mathbf{0}\delta^*\mathbf{1}$. Then given any two families $\{A_i \mid i \in J_m\}$ and $\{C_j \mid j \in J_n\}$ with $\mathbf{0} = \bigvee_i A_i$ and $\mathbf{1} = \bigvee_j C_j$, there exists a pair $(i,j) \in J_m \times J_n$ such that $A_i \mathcal{B} C_j$. Since $\mathbf{0} = \bigvee_i A_i$, $A_i = \mathbf{0}$ for all $i \in J_m$. So we take $D = \mathbf{0}$ and $E = \mathbf{1}$ such that $A_i \leq D$ and $C_j \leq E$. By (LFB3), $\mathbf{0}\mathcal{B}\mathbf{1}$. This is a contradiction. Thus $\mathbf{0} \not\delta^* \mathbf{1}$.

(LFP5) Suppose that $A \not\delta^* D$ and $C \not\delta^* D$. Then there exist families $\{A_i \mid i \in J_m\}$, $\{D_p \mid p \in J_k\}$, $\{C_j \mid j \in J_n\}$ and $\{E_q \mid q \in J_l\}$ with $A = \bigvee_i A_i$, $C = \bigvee_j C_j$ and $D = \bigvee_p D_p = \bigvee_q E_q$ such that $A_i \not\mathcal{B} D_p$ for every $(i,p) \in J_m \times J_k$ and $C_j \not\mathcal{B} E_q$ for every $(j,q) \in J_n \times J_l$. Let $A_{m+j} = C_j$ for $j \in J_n$. Consider $\{D_p \wedge E_q \mid p \in J_k, \ q \in J_l\}$ and $\{A_r \mid r \in J_{m+n}\}$. Then $D = \bigvee \{D_p \wedge E_q \mid p \in J_k, q \in J_l\}$ and $A \vee C = A_1 \vee A_2 \vee \cdots \vee A_m \vee A_{m+1} \vee \cdots \vee A_{m+n} = \bigvee_r A_r$. By (LFB3) and the above construction, we have $A_r \not\mathcal{B} (D_p \wedge E_q)$ for every $(p,q) \in J_k \times J_l$ and $r \in J_{m+n}$. Hence $(A \vee C) \not\mathcal{B} D$. Thus $(A \vee C) \not\delta^* D$.

(LFP6) Suppose $A \not\delta^* C$. Then there exist two families $\{A_i \mid i \in J_m\}$ and $\{C_j \mid j \in J_n\}$ with $A = \bigvee_i A_i$ and $C = \bigvee_j C_j$ such that $A_i \not\mathcal{B} C_j$ for every $(i,j) \in J_m \times J_n$. Since $A_i \not\mathcal{B} C_j$, by (LFB5), there exist $E_{ij} \in L^X$ for each $(i,j) \in J_m \times J_n$ such that $A_i \not\mathcal{B} E_{ij}$ and $E_{ij}' \not\mathcal{B} C_j$. Let $E_i = \bigvee_j E_{ij}$ and $E = \bigwedge_i E_i$. Then $A \not\delta^* E_i$ for every $i \in J_m$. So we have, by (LFP3), $A \not\delta^* E$. Since $E' = \bigvee_i E_i'$ and $E_i' = \bigwedge_j E_{ij}'$, we have $E_i' \not\mathcal{B} C_j$ by (LFB3). Hence we have $E' \not\delta^* C$.

Next, we show that $\delta^* \preceq \delta$ for any L-fuzzy proximity δ which is finer than $\mathcal{B}$. Let δ be any L-fuzzy proximity on X with $\mathcal{B} \preceq \delta$. Suppose

$A\delta C$. Let $\{A_i \mid i \in J_m\}$ and $\{C_j \mid j \in J_n\}$ be two families of elements of L^X with $A = \bigvee_i A_i$ and $C = \bigvee_j C_j$. Then, by (LFP5), there exist a pair $(i,j) \in J_m \times J_n$ such that $A_i \delta C_j$. Since $\mathcal{B} \preceq \delta$, we have $A_i \mathcal{B} C_j$. By the definition of δ^*, $A\delta^* C$. Hence $\delta^* \preceq \delta$. Thus δ^* is the coarsest L-fuzzy proximity on X which is finer than the L-fuzzy proximity base $\mathcal{B}$.

3. Main Part.

Now we prove the main results.

THEOREM 1. *The collection $\Delta(X)$ of all L-fuzzy proximities on a nonempty set X forms a complete lattice under the ordering $\preceq$.*

Proof. Let $\{\delta_\alpha \mid \alpha \in \Lambda\}$ be a nonempty collection of L-fuzzy proximities on a set X. Firstly, we show that there exists a coarsest L-fuzzy proximity $\delta(\equiv \sup_{\alpha \in \Lambda} \delta_\alpha)$ on X such that $\delta_\alpha \preceq \delta$ for every $\alpha \in \Lambda$. Let $\mathcal{B} = \bigcap\{\delta_\alpha \mid \alpha \in \Lambda\}$. Then since $\delta_\alpha \preceq \mathcal{B}$ for every $\alpha \in \Lambda$, $\mathcal{B}$ is an L-fuzzy proximity base. Therefore it generates an L-fuzzy proximity $\delta(\mathcal{B})$. Here we take $\delta = \delta(\mathcal{B})$. Then $\delta_\alpha \preceq \mathcal{B} \preceq \delta$. Thus δ is a coarsest L-fuzzy proximity on X such that $\delta_\alpha \preceq \delta$ for every $\alpha \in \Lambda$. Next, we show that there exists a finest L-fuzzy proximity $\delta(\equiv \inf_{\alpha \in \Lambda} \delta_\alpha)$ on X such that $\delta \preceq \delta_\alpha$ for every $\alpha \in \Lambda$. Let Ω be the collection of all L-fuzzy proximities on X which are coarser than δ_α for every $\alpha \in \Lambda$, i.e.,

$$\Omega = \{\delta^* \mid \delta^* \preceq \delta_\alpha \quad \text{for every} \quad \alpha \in \Lambda\}.$$

Since the indiscrete L-fuzzy proximity on X belongs to Ω, Ω is nonempty. Let

$$\delta = \sup\{\delta^* \mid \delta^* \in \Omega\}.$$

Let α be a fixed but arbitrary element of Λ, and $A\delta_\alpha C$. If $\{A_i \mid i \in J_m\}$ and $\{C_j \mid j \in J_n\}$ are arbitrary families of elements of L^X with $A = \bigvee_i A_i$ and $C = \bigvee_j C_j$, then there exists a pair $(i,j) \in J_m \times J_n$ such that $A_i \delta_\alpha C_j$. Since $\delta^* \preceq \delta_\alpha$ for every $\delta^* \in \Omega$, for the same pair $(i,j) \in J_m \times J_n$, $A_i \delta^* C_j$ for every $\delta^* \in \Omega$. Let $\mathcal{B} = \bigcap\{\delta^* \mid \delta^* \in \Omega\}$. Then $A_i \mathcal{B} C_j$. And since $\mathcal{B}$ is an L-fuzzy proximity base for δ and

$\delta \preceq \mathcal{B}$, we have $A\delta C$. Hence $\delta \preceq \delta_\alpha$ for every $\alpha \in \Lambda$. Moreover δ is finer than each member of Ω. Thus δ is the finest L-fuzzy proximity on X which is coarser than each member of the collection $\{\delta_\alpha \mid \alpha \in \Lambda\}$.

In the study of topological spaces, continuous functions play an important role. A similar role is played by L-proximity map in L-fuzzy proximity spaces.

DEFINITION 2. Let (X, δ_1) and (Y, δ_2) be two L-fuzzy proximity spaces. A function $f : X \to Y$ is said to be an *L-proximity map* iff for all $A, C \in L^X$,

$$A\delta_1 C \text{ implies } f[A]\delta_2 f[C].$$

Equivalently, f is an L-proximity map iff for all $D, E \in L^Y$,

$$D \not\delta_2 E \text{ implies } f^{-1}[D] \not\delta_1 f^{-1}[E].$$

There are important properties of the base of the L-fuzzy proximity.

THEOREM 2. *Let (X, δ_1) and (Y, δ_2) be two L-fuzzy proximity spaces. Let the L-fuzzy proximity δ_2 be generated by an L-fuzzy proximity base $\mathcal{B}$. Then the followings are equivalent:*

(a) A function $f : (X, \delta_1) \to (Y, \delta_2)$ is a L-proximity map

(b) $D \not\mathcal{B} E$ implies $f^{-1}[D] \not\delta_1 f^{-1}[E]$ for all $D, E \in L^Y$.

Proof. Let f be an L-fuzzy proximity map. Suppose $D \not\mathcal{B} E$ for all $D, E \in L^Y$. Since $\mathcal{B} \preceq \delta_2$, $D \not\delta_2 E$. Hence $f^{-1}[D] \not\delta_1 f^{-1}[E]$. Conversely, suppose that $D \not\mathcal{B} E$ implies $f^{-1}[D] \not\delta_1 f^{-1}[E]$ for all $D, E \in L^Y$. Let $D \not\delta_2 E$. If $D \not\mathcal{B} E$, then $f^{-1}[D] \not\delta_1 f^{-1}[E]$, trivially. If $D\mathcal{B}E$, then there exist two families $\{D_i \mid i \in J_m\}$ and $\{E_j \mid j \in J_n\}$ with $D = \bigvee_i D_i$ and $E = \bigvee_j E_j$ such that $D_i \not\mathcal{B} E_j$ for every $(i,j) \in J_m \times J_n$. Hence $f^{-1}[D_i] \not\delta_1 f^{-1}[E_j]$ for every $(i,j) \in J_m \times J_n$. So, $(\bigvee_i f^{-1}[D_i]) \not\delta_1 (\bigvee_j f^{-1}[E_j])$ by (LFP5). Therefore, since f^{-1} preserves arbitrary unions, $f^{-1}[D] \not\delta_1 f^{-1}[E]$.

THEOREM 3. *Let $f : X \to (Y, \delta_2)$ be a function. Then there exists the coarsest L-fuzzy proximity on X such that f is an L-proximity map.*

Proof. Let a binary relation $\mathcal{B}$ on L^X be defined as follows: $A\mathcal{B}C$ iff $f[A]\delta_2 f[C]$. Firstly, we show that $\mathcal{B}$ is an L-fuzzy proximity base on X.

(LFB1) If $A\mathcal{B}C$, $C\mathcal{B}A$ by (LFP1).

(LFB2) Suppose $A \not\mathcal{B} C$. Then $f[A]\delta_2 f[C]$. By (LFP2), $f[A] \leq (f[C])'$. Then $f^{-1}[f[A]] \leq f^{-1}[(f[C])']$. Since $A \leq f^{-1}[f[A]]$ and $f^{-1}[(f[C])'] = (f^{-1}[f[C]])' \leq C'$, $A \leq C'$.

(LFB3) Suppose $A\mathcal{B}C$, $A \leq D$ and $C \leq E$. Then $f[A]\delta_2 f[C]$, $f[A] \leq f[D]$ and $f[C] \leq f[E]$. By (LFP3), $f[D]\delta_2 f[E]$. Hence $D\mathcal{B}E$.

(LFB4) Let $\mathbf{0}_X$, $\mathbf{0}_Y$ and $\mathbf{1}_X$, $\mathbf{1}_Y$ are the least element and the greatest element of L^X and L^Y, respectively. Suppose $\mathbf{0}_X \mathcal{B} \mathbf{1}_X$. Then $f[\mathbf{0}_X]\delta_2 f[\mathbf{1}_X]$. Since $f[\mathbf{0}_X] = \mathbf{0}_Y$ and $f[\mathbf{1}_X] \leq \mathbf{1}_Y$, $\mathbf{0}_Y \delta_2 \mathbf{1}_Y$ by (LFP3). This is a contradiction. Hence $\mathbf{0}_X \not\mathcal{B} \mathbf{1}_X$.

(LFB5) Suppose $A \not\mathcal{B} C$. Then $f[A] \not\delta_2 f[C]$. By (LFP6) there exists $D \in L^Y$ such that $f[A] \not\delta_2 D$ and $D' \not\delta_2 f[C]$. Let $f^{-1}[D] = E \in L^X$. $f[E] = f[f^{-1}[D]] \leq D$ and by (LFP3) $f[A] \not\delta_2 f[E]$. Hence $A \not\mathcal{B} E$. Also $f[E'] = f[(f^{-1}[D])'] = f[f^{-1}[D']] \leq D'$ and by (LFP3) $f[E'] \not\delta_2 f[C]$. Hence $E' \not\mathcal{B} C$.

Therefore $\mathcal{B}$ is an L-fuzzy proximity base on X. From Lemma 1, there exists the coarsest L-fuzzy proximity $\delta = \delta(\mathcal{B})$ on X, which is finer than $\mathcal{B}$. Next, we show that $f : (X, \delta) \to (Y, \delta_2)$ is an L-proximity map. Suppose $f[A] \not\delta_2 f[C]$. Then $A \not\mathcal{B} C$. Since $\mathcal{B} \preceq \delta$, $A \not\delta C$. Hence f is an L-proximity map.

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