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NIRENBERG PROBLEMS ON 2–DIMENSIONAL DOMAINS

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I. INTRODUCTION

We are interested in the following problem raised by L. Nirenberg : What function K can be the Gaussian curvature of a metric g on a 2-dimensional manifold M, which is pointwise conformal to the standard metric g_0. Let $g = e^{2u}g_0$, where u is a smooth function on M. The problem is equivalent to the nonlinear partial differential equation

$$\Delta u + Ke^{2u} = k, \tag{1}$$

where Δ is the Laplace-Beltrami operator with respect to the metric g_0, and k is a constant Gaussian curvature of the metric g_0. Without loss of generality, we may take $k = 1(-1 \text{ or } 0)$.

In this paper, we consider a sufficient condition for the sovability of the equation (1) on the cup domain of the 2-dimensional sphere S^2 with nonnegative constant Dirichlet data or zero Neumann data. Also, we consider the zero Neumann boundary value problem on the domain D in a flat 2-dimensional torus T^2 which is enclosed by a set of geodesics.

The only known condition on curvature for compact 2-manifolds is the global condition given by the Gauss-Bonnet theorem :

$$\int_M K dS = 2\pi\chi(M),$$

where dS is the element of the area with respect to the metric g, and $\chi(M)$ is the Euler characteristic of M. This clearly imposes a sign condition K on M depending on $\chi(M)$.

We point out that some necessary conditions on the sphere S^2 are obvious for equation (1) to be solvable, namely,

(a) K should be positive somewhere,

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(b) Kadzan and Warner's implicit condition [1,4]

$$\int_{S^2} < \nabla K, \nabla x_i > e^{2u} dS = 0,$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Of course, the function $K(x) = 1 + \varepsilon x_1$ does not satisfy the hypothesis of this problem.

Moser[5] showed that the equation $\Delta u + Ke^{2u} = 1$ has an even solution on the sphere S^2 when K is an even function. His approach was to maximize a functional within the class of even functions in the Sobolev space H^1. The crucial point for his compactness argument is based on a sharp version of Trudinger's inequality ([1], see Proposition 2 below). Recently, Chang and Yang derive the corresponding version of inequality on the hemisphere. Using these inequalities, we have some sufficient conditions to the Nirenberg's problem on the various 2-dimensional domains.

II. Preliminaries

Throughout this paper, M will denote a compact, connected, standard 2-dimensional manifold. If M have a constant Gaussian curvature k with respect to the metric g_0, then ∇u and Δu will denote, respectively, the gradient and Laplacian of u, while

$$\overline{u} = \frac{1}{\text{vol}(M)} \int_M u d\mu = \fint u d\mu$$

will denote the average of u with respect to the element of area $d\mu$ determined by the metric g_0.

If D is a domain in M, then we use the same natation as above. Sobolev space $H_{s,p}(M)$ is the sets of functions on M whose derivatives up to order s are all in $L_p(M)$, we will write H^s instead of $H_{s,2}$.

We need the following well-known inequalities.

PROPOSITION 1(TRUDINGER[3,6]). *There are positive constants* β, γ *such that for any* $u \in H^1(M)$ *with* $\overline{u} = 0$ *and* $\|\nabla u\| \leq 1$, *one has* $\int_M e^{\beta u^2} d\mu \leq \gamma$.

Moser[5] has given an independent and quite different proof of Proposition 1 for the special cases $M = S^2$ and $M = P^2$, where P^2 is a 2-dimensional projective plane. He has shown that in both these cases the best constant is $\beta = 4\pi$.

PROPOSITION 2(MOSER). *For an even* C^1 *function* u *with* $\overline{u} = \int_{S^2} u d\mu = 0$ *and* $\int_{S^2} |\nabla u|^2 d\mu = 0$, *we have* $\int_{S^2} e^{8\pi u^2} d\mu \leq c_0$, *where* c_0 *is a univesal constant. Without an even condition, the inequality is weakened to*

$$\int_{S^2} e^{4\pi u^2} d\mu \leq c_0. \tag{2}$$

Recently, Chang and Yang [2] derive the corresponding version of Moser's inequality.

PROPOSITION 3(CHANG AND YANG). *Let* D *be a convex piecewise smooth* C^2 *domain in the plane with finite numbers of vertices. Let* θ_D *be the domain interior angle at the vertices of* D. *Then there exists a constant* C_D *such that for all*

$$u \in C^1(\overline{D}) \text{ with } \int_D |\nabla u|^2 d\mu \leq 1 \quad \text{and} \quad \int_D u d\mu = 0,$$

we have $\int_D e^{\beta u^2} d\mu \leq C_D$, *where* $\beta \leq 2\theta_D$.

As in the case of Moser's result, the integral is finite for all positive β, but if $\beta > 2\theta_D$, it can be made arbitrarily large by appropriate choice of u. If there is no corner on ∂D, then $\theta = \pi$ and $\beta = 2\pi$, which is the half index of Moser's theorem for Dirichlet boundary condition. They also obtain a corresponding version of this inequality when D is replaced by the lower hemisphere, then the exponent β may chosen to be 2π.

COROLLARY 1. *On the hemisphere* $H_0 = \{x \in S^2 \mid x_1^2 + x_2^2 + x_3^2 = 1,\ x_3 \leq 0\}$, *there is a constant* c_0 *such that for all* $u \in C^1(H_0)$,

$$\int_{H_0} u d\mu = 0 \quad \textit{and} \quad \int_{H_0} |\nabla u|^2 d\mu = 1,$$

we have

$$\int_{H_0} e^{2\pi u^2} d\mu \leq c_0. \tag{3}$$

This is also a direct consequence of Moser's inequality if we simply observed that such u may be reflected in the $x_1 x_2$ plane to be piecewise C^1 function.

For C^1 function u defined on H_0, let $\overline{u}$ denote $\frac{1}{2\pi}\int_{H_0} u d\mu = ⨍ u d\mu$, we have the following consequence of Corollary 1 based on the inequality

$$2(u - \overline{u}) \leq \frac{(u - \overline{u})^2}{⨍ |\nabla u|^2 d\mu} + ⨍ |\nabla u|^2 d\mu.$$

COROLLARY 2. *Suppose* $u \in C^1(H_0)$, *then there exists a constant* c_0 *such that*

$$\int_{H_0} e^{2u} d\mu \leq c_0 \exp\left(⨍ |\nabla u|^2 d\mu + ⨍ 2u d\mu \right).$$

3. MAIN THEROEMS

We consider the following Dirichlet boundary value problem on the cup domain

$$H_c = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^3 = 1, \quad x_3 \leq c, \quad -1 < c < 1\},$$

where c is a fixed constant,

$$(4)\qquad \begin{aligned} \Delta u + Ke^{2u} &= 1 \quad \text{in } H_c, \\ u \equiv \text{const.} &\geq 0 \quad \text{on } \partial H_c. \end{aligned}$$

Then we have the following result.

THEOREM 1. *Suppose that $K(\not\equiv 0)$ is a nonnegative smooth function on H_c, then equation (4) admits a smooth solution u.*

Proof. Without loss of generality, we may assume that $u \equiv 0$ on ∂H_c. We consider the functional

$$F(v) = \log \fint Ke^{2v} d\mu - \fint |\nabla v|^2 d\mu - 2\fint v d\mu,$$

where $d\mu$ is the area element corresponding to the metric $g_0 = dx_1^2 + dx_2^2 + dx_3^2$, and the integration is taken over the cup domain H_c. By the assumption on K, there exist smooth functions v on H_c, for which $\fint Ke^{2v} d\mu$ is positive and hence $F(v)$ is defined for those v. We will try to construct a solution to equation (4) from a function v which maximize the functional $F(v)$. Indeed, formally if v is such a function, the vanishing of the first variation of $F(v)$ yields

$$\frac{Ke^{2v}}{\fint Ke^{2v} d\mu} + \Delta v - 1 = 0.$$

These if we set $u = v + c$ and determine the constant c, so that $\fint Ke^{2u} d\mu = 1$, then u is a solution of (4).

This reduces the problem to finding a stationary point of $F(v)$ and we will search for a maximum. For the existence of a maximum, we need first that functional $F(v)$ is bounded from above and secondly that the supremum is taken on.

First we show that the functional is bounded from above. Because of zero Dirichlet data on the boundary of the cup domain H_c, we can extend v (resp. K) to the whole sphere S^2 by setting $v = 0$ (resp. $K = 0$) on $S^2 - H_c$, we still denote this extension function as u (resp. K).

Moser's inequality (Proposition 2) imposes the boundedness of F from above. Indeed, if v is a smooth function, set $v = aw + b$, where

$$b = \frac{1}{4\pi}\int_{S^2} v d\mu = \frac{1}{4\pi}\int_{H_c} v d\mu,$$

$$a^2 = \int_{S^2} |\nabla v|^2 d\mu = \int_{H_c} |\nabla v|^2 d\mu.$$

Estimating $2aw \le \frac{a^2}{\alpha} + \alpha w^2$, we obtain

$$\int_{H_c} Ke^{2v} d\mu = \int_{S^2} Ke^{2v} d\mu \le (\max K)\int_{S^2} \exp\left[\left(\frac{a^2}{\alpha}\right) + \alpha w^2 + 2b\right] d\mu.$$

Hence, by (2), $\log(⨍_{H_c} Ke^{2v} d\mu) \le \log\max K + \frac{a^2}{\alpha} + 2b + \log c_0$ and

$$\begin{aligned}F(v) &= \log(⨍_{H_c} Ke^{2v} d\mu) - \frac{a^2}{A} - 2⨍ v d\mu \\ &\le \log(c_0 \max K) + a^2\left(\frac{1}{\alpha} - \frac{1}{A}\right) + 2\left(\frac{1}{\alpha} - \frac{1}{A}\right)\int_{H_c} v d\mu,\end{aligned} \tag{5}$$

where A is the area of the cup domain H_c.

We consider the Hilbert space S obtained by completing the space of C^∞ functions $u = v(\xi)$ on S^2 satisfying $\int v d\xi = 0$, using the norm

$$\|u\| = \left(\int |\nabla v|^2 d\mu\right)^{\frac{1}{2}}.$$

We observe that $F(v + c) = F(v)$ for any constant c and therefore we can assume that $\int v d\xi = 0$ in the following.

Letting $v = v'\|v\|$, where $\|v'\| = 1$ with $\|v\|^2 = \int |\nabla v|^2 d\mu$ and noting that for every $\varepsilon > 0$, we have

$$2v \le \frac{1}{\varepsilon}\|v\|^2 + \varepsilon(v')^2.$$

We take $\varepsilon = 2\left(\frac{\alpha}{\alpha - A}\right)$ and we let

$$\delta = \left(\frac{1}{A} - \frac{1}{\alpha}\right) - \frac{1}{\varepsilon}A = \frac{1}{2}\left(\frac{1}{\alpha} - \frac{1}{A}\right).$$

We note that δ is always positive. By (5), we get

$$F(v) \leq \log(c_0 \max K) - \delta a^2 + \varepsilon. \tag{6}$$

Thus for $\alpha \geq 4\pi$, this gives $F(v) \leq \log(c_0 \max K)$. On the other hand, the inequality (2) holds for $\alpha \leq 4\pi$, so that we need it precisely for $\alpha = 4\pi$. Therefore, we have seen that the functional is indeed bounded from above.

Next we want to establish the existence of a maximum. Let v_n be a sequence of smooth functions in S such that $F(v_n) \to s = \sup_S F(v)$, where the supremum is taken on S.

We may assume that $F(v_n) > s - 1$, so that by (6),

$$\int |\nabla v_n|^2 d\mu \leq C_1 = \frac{1}{\delta}(1 - s + \log(c_0 \max K) + \varepsilon), \tag{7}$$

Since the closed ball in the Hilbert space is weakly compact, we can select a subsequence, called v_n again, such that v_n converges weakly to an element $\tilde{v}$ in S.

If we can show that $⨍ K e^{2v} d\mu$ is continuous in the weak topology on S, it follows easily that $F(\tilde{v}) = s$, namely, the supremum is taken on H_c. the continuity of $⨍ K e^{2v} d\mu$ is an consequence of Rellich's compactness theorem [1]. Since $\tilde{v}$ is a maximum for $F(v)$, the function $u = \tilde{v} + c$ with an appropriate constant c is a weak solution of (4).

Standard regularity theorem [1,3] show that $u(\xi)$ is a smooth function as K is. This concludes the existence proof of theorem. □

We will consider the following Neumann boundary value problem on the cup domain H_c with $-1 < c < 0$,

$$\begin{aligned} \Delta u + K e^{2u} &= 1 \quad \text{in } H_c, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } H_c, \end{aligned} \tag{8}$$

where $\dfrac{\partial u}{\partial n}$ denote the unit normal derivative of u. The boundary condition means that geometrically $\{x_3 = c\}$ remains a geodesic in the new conformal metric $g = e^{2u} g_0$.

Chang and Yang [2] get the following sufficient condition for the conformal deformation equation the hemisphere, namely $c = 0$, with zero Neumann data.

THEOREM. *Suppose that K is a smooth function in H_0 with*

$$⨍ K d\mu > \max_{x \in \partial H_0} K, \quad ⨍ K d\mu > 0.$$

Then equation (8) admits a smooth solution u.

In the cup domain, we easily have the following theorem.

THEOREM 2. *Suppose that K is a smooth function in H_c $(-1 < c < 0)$ with $⨍ K d\mu > 0$, then equation (8) has a smooth solution u.*

Proof. The proof is quite similar to that of Theorem 1.

We consider the functional

$$F(u) = \log ⨍ K e^{2u} d\mu - (⨍ |\nabla u|^2 d\mu + 2 ⨍ u d\mu).$$

We note that $⨍ K e^{2u} d\mu > 0$. We will obtain a solution of (8) by maximizing the functional $F(u)$ subject to the constraint $⨍ u d\mu = 0$. Let $S = \{u \in H^1(H_c) \mid f u d\mu = 0\}$ be a Hilbert space. As before, the Euler-Lagrange equation for the critical points of F (if they exist) give (after modifying by additive constants) exactly (8).

We begin by noting that as a consequence of Corollary 2, the functional $F(u)$ is bounded above. Thus we can take a subsequence u_α in $H^1(H_c)$ with $\lim_{\alpha \to \infty} F(u_\alpha) = \sup_S F(u) = s$. Since H_c is compact, there exist a maximum $\|K\|_\infty$,

$$\begin{aligned} F(u) &\le \log\|K\|_\infty + \log \fint e^{2u} d\mu - \fint |\nabla u|^2 d\mu - 2\fint u d\mu \\ &\le \log\|K\|_\infty + \log \fint \exp\left(2\pi \frac{u^2}{\int |\nabla u|^2 d\mu} + \frac{1}{2\pi}\int |\nabla u|^2 d\mu\right) d\mu \\ &\quad - \fint |\nabla u|^2 d\mu - 2\fint u d\mu \\ &\le \log(c_0\|K\|_\infty) + \left(\frac{1}{2\pi} - \frac{1}{A}\right)\int |\nabla u|^2 d\mu - 2\fint u d\mu, \end{aligned}$$

where A is the area of H_c.

We remark that $\fint \exp\left(\frac{u^2}{\int |\nabla u|^2 du}\right) d\mu \le c_0$, where c_0 is a universal constant. We note that $\frac{1}{2}\left(\frac{1}{A} - \frac{1}{2\pi}\right) = \delta$ is always positive.

We may assume that $F(u_\alpha) > s - 1$. Thus

$$\fint |\nabla u|^2 d\mu \le \frac{1}{\delta}\left(\log(c_0\|K\|_\infty) + 1 - s + \varepsilon\right).$$

Hence, we can extract a weakly convergent subsequence u_α in $H^1(H_c)$ whose limit, by the same argument as in Theorem 1, is a weak solution of (8). Then the usual regularity theory for Laplace equation shows that u is a C^∞ solution of (8). □

Let T^2 be a flat 2-dimensional torus and let Ω be a collection of geodesics $\{r_j\}$. The domain D in T^2 is enclosed by the set of geodesics, namely, $\partial D = \bigcup_{r_j \in \Omega} r_j$.

We will finally consider the Neumann boundary value problem

$$\begin{aligned} \Delta u + K e^{2u} &= 0 \quad \text{in } D, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D. \end{aligned} \tag{9}$$

Then we have the following.

THEOREM 3. *Let $K(\not\equiv 0)$ belong to $C^\infty(D)$. Then the conditions*

(i) *K changes sign,*

(ii) *$\int_D K d\mu < 0$*

are necessary and sufficient for the existence of a solution $u \in C^\infty(D)$ of equation (9). Futhermore, there exists a unique solution.

Proof. The proof of the necessary and sufficient conditions are almost the same as that of Kadzan and Warner [4]. Therefore we only establish the uniqueness of solution.

Let

$$S = \{u \in H^1(D) \mid \int_D Ke^{2v} d\mu = 0, \quad \overline{v} = \int_D v d\mu = 0\}.$$

Let $u_t = u + tv \in S$. From $0 = \int_D Ke^{2u_t} d\mu = \int_D Ke^{2u}(1 + tv + \cdots) d\mu$, we get $\int_D vKe^{2u} d\mu = 0$. Computing the second variation of $F(u) = \int_D |\nabla u|^2 d\mu$, then we have

$$\begin{aligned} \frac{d^2}{dt^2} \int_D |\nabla u_t|^2 d\mu &= \frac{d^2}{dt^2} \left(\int_D (|\nabla u|^2 + 2\nabla u \cdot \nabla v + t^2 |\nabla u|^2 + \cdots) d\mu \right) \\ &= 2 \int_D |\nabla u|^2 d\mu \geq 0. \end{aligned}$$

Therefore the critical points of $F(u)$ are local minima.

Suppose that there are two critical points such that one point is the minimum, namely, there are two solutions u and v. Then we can find a third solution which is not a local minimum. We can take a curve $u_t(t \in [0,1]) \in S$, where $u_0 = u, u_1 = v$. Consider the functional $\max F(u_t)$, we know that the compactness argument still holds in this functional. Therefore we can minimize $\max_t F(u_t)$. There is an element $\tilde{u}$ such that $F(\tilde{u}) = \min_S \max_t F(u_t)$. Such an element cannot be a local minimum, by the maximality condition. It is an obvious contradiciton. □

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