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SOME CONGRUENCES FOR BERNOULLI NUMBERS

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§ 1. Introduction

Throughout this paper $\mathbf{Z}_p$, $\mathbf{Q}_p$ and $\mathbf{C}_p$ will respectively denote the ring of p–adic rational integers, the field of p–adic rational numbers and the completion of the algebraic closure of $\mathbf{Q}_p$.

Let $C(\mathbf{Z}_p, \mathbf{C}_p)$ and $UD(\mathbf{Z}_p, \mathbf{C}_p)$ denote the space of all continuous functions and the space of all uniformly differentiable functions on $\mathbf{Z}_p$ with values in $\mathbf{C}_p$. For $f \in UD(\mathbf{Z}_p, \mathbf{C}_p)$, we have an integral $I_0(f)$ with respect to use so called invariant measure μ_0;

$$I_0(f) = \int_{\mathbf{Z}_p} f(x) d\mu_0(x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{x=0}^{p^n - 1} f(x),$$

where $\mu_0(x + p^n \mathbf{Z}_p) = \frac{1}{p^n}$.

Let C_{p^n} be the cyclic group consisting of all p^n–th roots of unity in $\mathbf{C}_p$ for all $n \geq 0$ and C_{p^∞} be the direct limit of C_{p^n} with respect to the natural morphisms, hence C_{p^∞} is the union of all C_{p^n} with discrete topology.

We shall consider various space H derived from $\mathbf{Q}_p$–valued continuous functions on $\mathbf{Z}_p$ on which $\mathbf{Z}_p$ will act in the way (induced by translation), $n \longmapsto n_x$ for $\mathbf{Z}_p$.

Let $H^{\mathbf{Z}_p} = \{n \in H | n = n_x \quad \text{for} \quad x \in \quad \mathbf{Z}_p\}$. Here v_p will denote the normalized exponential valuation of $\mathbf{C}_p$ and let $Char(p^n \mathbf{Z}_p)$ denote the characteristic function of $p^n \mathbf{Z}_p$ $(n \geq 0)$.

In this paper, we will give some properties on
$\mathrm{Hom}_{\mathbf{Z}_p}(UD(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p}$.
By this properties, we immediately deduce the "Kummer congruences" for the Bernoulli numbers.

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§ 2. Some properties of p-adic integral on $\mathbf{Z}_p$

Let M be topological $\mathbf{Z}_p$-module. For $K \in \mathrm{Hom}_{\mathbf{Z}_p}(C(\mathbf{Z}_p, \mathbf{C}_p), M)$, we define the operator K^x $(x \in \mathbf{Z}_p)$ by

$$K^x(f) = K(f_x).$$

It is well known that

$$\binom{x+1}{n+1} = \binom{x}{n+1} + \binom{x}{n} \quad \text{for} \quad x \in \mathbf{Z}_p.$$

Thus

$$K(\binom{x+1}{n+1}) = K(\binom{x}{n+1}) + K(\binom{x}{n}),$$

where K is invariant operator.
Hence $K(\binom{x}{n}) = 0$ because of $K(\binom{x+1}{n+1}) = K^1(\binom{x}{n+1}) = K(\binom{x}{n+1})$.

For $f \in C(\mathbf{Z}_p, \mathbf{C}_p)$, the Mahler's expansion is defined by

$$f(x) = \sum_{n=0}^{\infty} \Delta^n f(0) \binom{x}{n} \quad \text{for all} \quad x \in \mathbf{Z}_p,$$

where $\Delta f(x) = f(x+1) - f(x)$.
Thus we have

$$K(f) = \lim_{l \to \infty} \sum_{n=0}^{l} \Delta^n f(0) K(\binom{x}{n}) = 0.$$

Therefore we obtain the following;

$$\mathrm{Hom}_{\mathbf{Z}_p}(C(\mathbf{Z}_p, \mathbf{Z}_p), M)^{\mathbf{Z}_p} = 0 \quad \text{for} \quad f \in C(\mathbf{Z}_p, \mathbf{C}_p).$$

Let $M = \mathbf{Q}_p/\mathbf{Z}_p$. Then $\mathrm{Hom}_{\mathbf{Z}_p}(C(\mathbf{Z}_p, \mathbf{Z}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p} = 0$.
We denote by

$$UD_1(\mathbf{Z}_p, \mathbf{Q}_p) = \{f \in UD(\mathbf{Z}_p, \mathbf{Q}_p) | f' \in C(\mathbf{Z}_p, \mathbf{C}_p)\}$$
$$\mathrm{Int}(\mathbf{Z}_p, \mathbf{C}_p) = \{f \in UD(\mathbf{Z}_p, \mathbf{Q}_p) | f' = 0\}.$$

It was known in [2][3] that

$$0 \to \mathrm{Int}(\mathbf{Z}_p, \mathbf{Q}_p) \to UD_1(\mathbf{Z}_p, \mathbf{Q}_p) \to C(\mathbf{Z}_p, \mathbf{Z}_p) \to 0$$

: exact sequence (Dieudonne Theorem).

Thus we have

$$0 \to \mathrm{Hom}_{\mathbf{Z}_p}(C(\mathbf{Z}_p, \mathbf{Z}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p} \to \mathrm{Hom}_{\mathbf{Z}_p}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p}$$
$$\to \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Int}(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p} \cong \mathbf{Z}_p \to 0$$

: exact sequence.

LEMMA 1. *Let* $X = \{f | f : \mathbf{Z}_p, \to \mathbf{Q}_p :$ *locally constant function* $\}$. *Then* $\bar{X} =$ *Int*$(\mathbf{Z}_p, \mathbf{Q}_p)$, *where* $\bar{X}$ *is closure of* X *in* $UD(\mathbf{Z}_p, \mathbf{Q}_p)$.

This lemma can be found in [3].

Proof. It is easy to see in [3] that $\bar{X} \subset \mathrm{Int}(\mathbf{Z}_p, \mathbf{Q}_p)$.

Now, we set $x_n \equiv x \pmod{p^n}$, $0 \leq x_n \leq p^n - 1$ $(n = 0, 1, 2, \cdots)$.
Let $f_n(x) = f(x_n)$.
For $f \in \mathrm{Int}(\mathbf{Z}_p, \mathbf{Q}_p)$, we have $f_n(x) = f(x_n) = f(x - p^n[\frac{x}{p^n}])$, where $[\cdot]$ is the Gauss' symbol. Thus $f_n \in X$.
From the definition of valuation, we see that

$$v(f - f_n) = \mathrm{Inf}_{x \in \mathbf{Z}_p} v_p(f(x) - f(x_n)) \geq R(f) + n,$$

where

$$R(f - f_n) \overset{\mathrm{def}}{\equiv} \inf_{\substack{x, y \in \mathbf{Z}_p \\ x \neq y}} v_p\left(\frac{f(x) - f(x_n) - f(y) + f(y_n)}{x - y}\right).$$

Thus $v(f - f_n) \to \infty$ (as $n \to \infty$).
Therefore $f = \lim_{n \to \infty} f_n \in \bar{X}$. (i.e. $\bar{X} = \mathrm{Int}(\mathbf{Z}_p, \mathbf{Q}_p)$.)

PROPOSITION 1. *Let* $J : UD_1(\mathbf{Z}_p, \mathbf{Q}_p) \xrightarrow{I_0} \mathbf{Q}_p \xrightarrow{\mathcal{N}} \mathbf{Q}_p/\mathbf{Z}_p$. *Then* $J \in Hom_{\mathbf{Z}_p}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p}$.

Proof. It was known in [3] that $I_0(f+g) = I_0(f) + I_0(g)$.
By definition of J^x, we have

$$\begin{aligned} J^x(f) &= J(f_x) = \mathcal{N}(I_0(f_x)) \\ &= \mathcal{N}(I_0(f) + \lim_{n\to x} \sum_{i=0}^{n-1} f'(i)) = J(f) \end{aligned}$$

for all $x \in \mathbf{Z}_p$ and for all $f \in UD_1(\mathbf{Z}_p, \mathbf{Q}_p)$,
because of $I_0(f_n) = I_0(f) + \sum_{i=0}^{n-1} f'(i)$, $f' \in C(\mathbf{Z}_p, \mathbf{Z}_p)$, where $f_n(x) = f(x+n)$. Hence $\mathcal{N}(f'(x)) = 0$.

PROPOSITION 2. $\mathbf{Z}_p \cdot J \subset Hom_{\mathbf{Z}_p}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p}$.

Proof. By the above proposition, $J \in \mathrm{Hom}_{\mathbf{Z}_p}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)$.
Hence

$$x \cdot J(f) = J(f_x) = J^x(f) \in \mathrm{Hom}_{\mathbf{Z}_p}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p}.$$

Thus

$$\mathbf{Z}_p \cdot J \subset \mathrm{Hom}_{\mathbf{Z}_p}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p}.$$

Let $K \in \mathrm{Hom}_{\mathbf{Z}_p}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p}$.
Then $K(x+1) = K(x) + K(1), K(1) = 0$.

Now, we set $\psi_{0,n}(x) = Char(p^n\mathbf{Z}_p)$ and $\psi_{a,n}(x) = Char(a + p^n\mathbf{Z}_p)$.
Thus $p^n K(\psi_{0,n}) = \sum_{a=0}^{p^n-1} K(\psi_{a,n}) = K(1) = 0$.
Hence

$$\begin{aligned} K(\psi_{0,n}) &\in \frac{1}{p^n}\mathbf{Z}_p/\mathbf{Z}_p \cong C_{p^n}, \\ J(\psi_{0,n}) &= \frac{1}{p^n} \pmod{p^0}, \end{aligned}$$

because of $\frac{1}{p^n} = I_0(\psi_{0,n}) = \int_{p^n\mathbf{Z}_p} d\mu_0(x)$.

Let $K(\psi_{0,n}) = \sigma_n (n \geq 0)$. Then $p\sigma_{n+1} = \sigma_n$.

Now, we set $\omega_n = \mathcal{N}(\frac{1}{p^n}) = \frac{1}{p^n} \pmod{p^0}$. Thus $\omega_n = J(\psi_{0,n})$.
There exists $\alpha \in \mathbf{Z}_p$ such that $\alpha(a\omega_n) = a\sigma_n$ for all $a \in \mathbf{Z}_p$,
since $\alpha(a\omega_n) = a\alpha(\omega_n) = a(\frac{\alpha}{p^n}) = a(K(\psi_{0,n})) = a\sigma_n$.

Thus $K - \alpha J = 0$. Hence $K = \alpha J \in \mathbf{Z}_p \cdot J$.

Therefore we obtain the following;

THEOREM 1. *$Hom_{\mathbf{Z}_p}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p} \cong \mathbf{Z}_p \cdot J$.*

§ 3. Kummer Congruences

For $u \in \mathbf{Z}_p^\times$, we define J^u by $J^u(f(x)) = J(f(ux))$.
Then $J(f) = J^u(f)$ for $f \in UD_1(\mathbf{Z}_p, \mathbf{Q}_p)$.
If $f(x) = \psi_{0,n}(x)$, then $J^u(\psi_{0,n}(x)) = J(\psi_{0,n}(x))$. Hence $J \cdot J^u \in \mathrm{Hom}(UD_1(\mathbf{Z}_p, \mathbf{Q}_p), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathbf{Z}_p}$.

For any sequence $\{a_k\}$, we define an operator Δ and Δ_k by $\Delta a_k = a_{k+1} - a_k$ and $\Delta_k = (1+\Delta)^k$.
Let $0 \le l \le n-1$ with $p-1 \nmid n$. Here, we set

$$f(x) = \frac{1}{p^l}\Delta_{p-1}^l \frac{1}{n}x^n.$$

Then $f(x) \in UD_1(\mathbf{Z}_p, \mathbf{Q}_p)$, since

$$\begin{aligned} f'(x) &= \frac{1}{p^l}\Big(\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}\frac{1}{n+(p-1)i}x^{n+(p-1)i}\Big)' \\ &= x^{n-1}\Big(\frac{x^{p-1}-1}{p}\Big)^l \in C(\mathbf{Z}_p, \mathbf{Z}_p). \end{aligned}$$

It is easy to see that

$$J^u\Big(\frac{1}{p^l}\Delta_{p-1}^l\frac{1}{n}x^n\Big) = J\Big(\frac{1}{p^l}\Delta_{p-1}^l\frac{1}{n}x^n\Big).$$

Thus $(u^n - 1)I_0(\frac{1}{p^l}\Delta_{p-1}^l\frac{1}{n}x^n) \equiv 0 \pmod{p^0}$

It is well known in [2][3][4][5][6] that

$$\int_{\mathbf{Z}_p} x^n d\mu_0(x) = I_0(x^n) = B_n,$$

where B_n is n–th Bernoulli number.
In particular, we take $u = \zeta_{p-1}$, $\zeta_{p-1} - 1 \not\equiv 0 \pmod{p^0}$.
Then $\Delta^l_{p-1}\frac{B_n}{n} \equiv 0 \pmod{p^l}$.
Therefore we obtain Kummer congruence;

$$\frac{1}{n}B_n \equiv \frac{1}{n+p-1}B_{n+p-1} \pmod{p}.$$

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