

CONFORMAL CHANGE OF THE TENSOR $U^\nu{}_{\lambda\mu}$ IN 6-DIMENSIONAL g -UFT

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I. INTRODUCTION.

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ([8], 1957). CHUNG ([6], 1968) also investigated the same topic in 4-dimensional *g -unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional case were investigated by CHO([1], 1992), ([2], 1994). CHO([3], 1995) also studied change of the torsion tensor $S_{w\mu}{}^\nu$ induced by the conformal change for the second class with the first category in 6-dimensional g -unified field theory.

In the present paper, we investigate change of the tensor $U^\nu{}_{\lambda\mu}$ induced by the conformal change in 6-dimensional g -unified field theory. These topics will be studied for the second class with the first category in 6-dimensional case.

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II. PRELIMINARIES.

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be referred to CHUNG([5], 1982; [4], 1988), CHO([1], 1992; [2], 1994; [3], 1995).

2.1. n -dimensional g -unified field theory.

The n -dimensional g -unified field theory (n - g -UFT hereafter) was originally suggested by HLAVATÝ([8], 1957) and systematically introduced by CHUNG([7], 1963).

Let X_n ¹ be an n -dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \rightarrow x^{\nu'}$, for which

$$(2.1) \quad \text{Det} \left(\left(\frac{\partial x}{\partial x'} \right) \right) \neq 0.$$

In the usual Einstein's n -dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ ² :

$$(2.2) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

where

$$(2.3) \quad \text{Det}((g_{\lambda\mu})) \neq 0, \quad \text{Det}((h_{\lambda\mu})) \neq 0.$$

¹Throughout the present paper, we assumed that $n \geq 2$.

²Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \dots, n$ and follow the summation convention.

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$(2.4) \quad h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu.$$

In our n - g -UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma_{\omega\mu}^\nu$ with the following transformation rule :

$$(2.5) \quad \Gamma_{\omega'\mu'}^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma_{\beta\gamma}^\alpha + \frac{\partial^2 x^\alpha}{\partial x^{\omega'} \partial x^{\mu'}} \right)$$

and satisfies the system of Einstein's equations

$$(2.6) \quad D_w g_{\lambda\mu} = 2S_{w\mu}{}^\alpha g_{\lambda\alpha}$$

where D_w denotes the covariant derivative with respect to $\Gamma_{\lambda\mu}^\nu$ and

$$(2.7) \quad S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda\mu]}^\nu$$

is the *torsion tensor* of $\Gamma_{\lambda\mu}^\nu$. The connection $\Gamma_{\lambda\mu}^\nu$ satisfying (2.6) is called the *Einstein's connection*.

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \dots$ are frequently used :

$$(2.8)a \quad \mathfrak{g} = \text{Det}((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = \text{Det}((h_{\lambda\mu})) \neq 0, \\ \mathfrak{k} = \text{Det}((k_{\lambda\mu})),$$

$$(2.8)b \quad g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}},$$

$$(2.8)c \quad K_p = k_{[\alpha_1}{}^{\alpha_1} \dots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \dots)$$

$$(2.8)d \quad {}^{(0)}k_\lambda{}^\nu = \delta_\lambda^\nu, \quad {}^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \quad {}^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu,$$

$$(2.8)e \quad K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu},$$

$$(2.8)f \quad \sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\{\overset{\nu}{\lambda}_\mu\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

$$(2.9)a \quad K_0 = 1; \quad K_n = k \text{ if } n \text{ is even; } K_p = 0 \text{ if } p \text{ is odd,}$$

$$(2.9)b \quad g = 1 + K_2 + \cdots + K_{n-\sigma},$$

$$(2.9)c \quad {}^{(p)}k_{\lambda\mu} = (-1)^{p(p)} k_{\mu\lambda}, \quad {}^{(p)}k^{\lambda\nu} = (-1)^{p(p)} k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T :

$$(2.10)a \quad \overset{pqr}{T} = \overset{pqr}{T}_{\omega\mu\nu} = T_{\alpha\beta\gamma} {}^{(p)}k_\omega{}^{\alpha(q)} k_\mu{}^{\beta(r)} k_\nu{}^\gamma,$$

$$(2.10)b \quad T = T_{\omega\mu\nu} = \overset{000}{T},$$

$$(2.10)c \quad 2 \overset{pqr}{T}_{\omega[\lambda\mu]} = \overset{pqr}{T}_{\omega\lambda\mu} - \overset{pqr}{T}_{\omega\mu\lambda},$$

$$(2.10)d \quad 2 \overset{(pq)r}{T}_{\omega\lambda\mu} = \overset{pqr}{T}_{\omega\lambda\mu} + \overset{qpr}{T}_{\omega\lambda\mu}.$$

We then have

$$(2.11) \quad \overset{pqr}{T}_{\omega\lambda\mu} = - \overset{qpr}{T}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma^\nu_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

$$(2.12) \quad \Gamma^\nu_{\omega\mu} = \{\nu_{\omega\mu}\} + S_{\omega\mu}{}^\nu + U^\nu_{\omega\mu}$$

where

$$(2.13) \quad U_{\nu\omega\mu} = S^{100}_{(\omega\mu)\nu} + S^{(10)0}_{\nu(\omega\mu)}.$$

The above two relations show that our problem of determining $\Gamma^\nu_{\omega\mu}$ in terms of $g_{\lambda\mu}$ is reduced to that of studying the tensor $S_{\omega\mu}{}^\nu$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^\nu$ satisfies

$$(2.14) \quad S = B - 3 S^{(110)}$$

where

$$(2.15) \quad 2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k_{\omega]}{}^\alpha k_\nu{}^\beta.$$

2.2. Some results in 6- g -UFT.

In this section, we introduce some results of 6- g -UFT without proof, which are needed in our subsequent considerations.

DEFINITION (2.1). In 6- g -UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be of the second class with the first category, if $K_2 \neq 0$, $K_4 = K_6 = 0$.

THEOREM (2.2). (Main recurrence relation) In X_6 , the following recurrence relation holds :

(Second class with the first category)

$$(2.16) \quad {}^{(p+2)}k_\lambda{}^\nu = -K_2 {}^{(p)}k_\lambda{}^\nu, \quad (p = 1, 2, \dots)$$

THEOREM (2.3). (For the second class with the first category in 6-g-UFFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is

$$(2.17) \quad 1 - (K_2)^2 \neq 0.$$

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

$$(2.18) \quad (1 - (K_2)^2)(B - S) = K_2(1 - K_2)B + 2 \overset{(10)1}{B}.$$

III. CONFORMAL CHANGE OF THE 6-DIMENSIONAL TENSOR $U^\nu{}_{\lambda\mu}$ FOR THE SECOND CLASS WITH THE FIRST CATEGORY.

In this final chapter we investigate the change $U^\nu{}_{\lambda\mu} \rightarrow \overline{U}^\nu{}_{\lambda\mu}$ of the tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and \overline{X}_n are conformal if and only if

$$(3.1) \quad \overline{g}_{\lambda\mu}(x) = e^\Omega g_{\lambda\mu}(x)$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the tensor $U^\nu{}_{\lambda\mu}$. An explicit representation of the change of 6-dimensional tensor $U^\nu{}_{\lambda\mu}$ for the second class with the first category will be exhibited in this chapter.

AGREEMENT (3.1). Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote \overline{T} the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is \overline{T} . Furthermore, the indices of T (\overline{T}) will be raised and/or lowered by means of $h^{\lambda\nu}$ ($\overline{h}^{\lambda\nu}$) and/or $h_{\lambda\mu}$ ($\overline{h}_{\lambda\mu}$).

The results in the following theorems are needed in our further considerations. They may be referred to CHO([1], 1992; [2], 1994; [3], 1995).

THEOREM (3.2). In n - g -UFT, the conformal change (3.1) induces the following changes :

$$(3.2)a \quad \begin{aligned} {}^{(p)}\bar{k}_{\lambda\mu} &= e^{\Omega(p)} k_{\lambda\mu}, & {}^{(p)}\bar{k}_\lambda{}^\nu &= {}^{(p)}k_\lambda{}^\nu, \\ {}^{(p)}\bar{k}^{\lambda\nu} &= e^{-\Omega(p)} k^{\lambda\nu} \end{aligned}$$

$$(3.2)b \quad \bar{g} = g, \quad \bar{K}_p = K_p, \quad (p = 1, 2, \dots).$$

Now, we are ready to derive representations of the changes $U^\nu{}_{\lambda\mu} \rightarrow \bar{U}^\nu{}_{\lambda\mu}$ in 6- g -UFT for the second class with the first category induced by the conformal change (3.1).

THEOREM (3.3). (For the second class with the first category) The change $S_{w\mu}{}^\nu \rightarrow \bar{S}_{w\mu}{}^\nu$ induced by conformal change (3.1) may be represented by

$$(3.3) \quad \begin{aligned} \bar{S}_{w\mu}{}^\nu &= S_{w\mu}{}^\nu + \frac{1}{1 - (K_2)^2} (-4K_2^{(2)} k^\nu{}_{[w} k_{\mu]}{}^\delta \Omega_\delta \\ &+ (1 - 2K_2) k^\nu{}_{[w} \Omega_{\mu]} + (1 + 2K_2) k^\nu{}_{[w} {}^{(2)}k_{\mu]}{}^\delta \Omega_\delta) \\ &+ \frac{1}{1 + K_2} (-k_{w\mu} \Omega^\nu - h^\nu{}_{[w} k_{\mu]}{}^\delta \Omega_\delta + k_{w\mu} {}^{(2)}k^{\nu\delta} \Omega_\delta), \end{aligned}$$

where $\Omega_\mu = \partial_\mu \Omega$.

THEOREM (3.4). The change $U^\nu{}_{\lambda\mu} \rightarrow \bar{U}^\nu{}_{\lambda\mu}$ induced by the conformal change (3.1) may be represented by

$$(3.4) \quad \begin{aligned} \bar{U}^\nu{}_{\lambda\mu} &= U^\nu{}_{\lambda\mu} + \frac{1}{1 - (K_2)^2} \left\{ (2 - 9K_2) {}^{(2)}k^\nu{}_{(\lambda} {}^{(2)}k_{\mu)}{}^\delta \right. \\ &+ (3 + K_2) {}^{(2)}k_{\lambda\mu} {}^{(2)}k^{\nu\delta} + 2(2(K_2)^2 - 1) k^\nu{}_{(\lambda} k_{\mu)}{}^\delta \left. \right\} \Omega_\delta \\ &+ \frac{1}{1 - K_2} \left\{ 2 {}^{(2)}k^\nu{}_{(\lambda} \Omega_{\mu)} - \frac{1 + 4K_2}{1 + K_2} {}^{(2)}k_{\lambda\mu} \Omega^\nu \right. \\ &\quad \left. - 2k^\nu{}_{(\lambda} {}^{(2)}k_{\mu)}{}^\delta \Omega_\delta \right\} + \frac{1}{1 + K_2} h_{\lambda\mu} {}^{(2)}k^{\nu\delta} \Omega_\delta. \end{aligned}$$

Proof. In virtue of (2.13) and Agreement (3.1), we have

$$(3.5) \quad \overline{U}_{\nu\lambda\mu} = \overline{S}_{(\lambda\mu)\nu}^{\overline{100}} + \overline{S}_{\nu(\lambda\mu)}^{\overline{(10)0}}.$$

The relation (3.4) follows by substituiting (3.3), (2.10), Definition (2.1), (2.16), (3.2) into (3.5). \square

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