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TENT SPACES OVER LIPSCHITZ DOMAINS WITH APPROACH REGIONS

BYUNG–OH PARK

1. Introduction

Several authors have studied the $\mathbf{L}^p$ boundedness of maximal functions defined by means of general subsets. This depends on an atomic decomposition for certain tent spaces. This was proved in the Euclidean case $\mathbf{R}_+^{n+1}$ by Coifman, Meyer, and Stein[2]. Also, María J,Carro and Javier Soria have studied the tent spaces over general approach regions and their atomic decomposition.

In this paper, we are going to define a tent spaces over Lipschitz domains with approach regions. Also, duality and atomic decomposition of tent spaces generalize the earlier results.([1],[2])

This purpose of the present paper is to show that every element of the tent spaces $\mathbf{T}_\Omega^p(\mathcal{L})(0<\text{p}\leq 1)$ can be decomposed into particles which are called "atoms"[Thm 1] and the dual space of $\mathbf{T}_\Omega^p(\mathcal{L})(0<\text{p}\leq 1)$ is the space of Carleson measure[Thm 2].

2. Preliminaries

A real valued function ϕ defined on R^n is said to be a *Lipschitz function* if there exists a constant M such that $|\phi(x)-\phi(y)| \leq M|x-y|$ for all $x, y \in R^n$.

Let $\mathcal{L}$ be the set

$$\mathcal{L} = \{(y,t) \in R^n \times R : \phi(y) < t\}$$

Then $\mathcal{L}$ is called a *Lipschitz domain* determined by ϕ. The boundary of $\mathcal{L}$ will be denoted by $\partial\mathcal{L}$. For $\widetilde{x} = (x, \phi(x)) \in \partial\mathcal{L}$, let π be the projection of $\partial\mathcal{L}$ onto R^n given by $\pi(\widetilde{x}) = x$. A set $U \subset \partial\mathcal{L}$ is said to be *open* if $\pi(U)$ is open in R^n. Also, we will denote ds by the area measure on $\partial\mathcal{L}$.

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Let $\Omega = \{\Omega_{\tilde{x}}\}_{\tilde{x}\in\partial\mathcal{L}}$ be a collection of measurable subset, where $\Omega_{\tilde{x}} \subset \mathcal{L}$. For a measurable function f on $\mathcal{L}$. We define the *maximal function* of f with respect to Ω as

$$\mathcal{A}_{\Omega}^{\infty}(f)(\tilde{x}) = \sup_{(y,t)\in\Omega_{\tilde{x}}} |f(y,t)|.$$

We will always assume that Ω is choosen so that $\mathcal{A}_{\Omega}^{\infty}(f)$ is a measurable function. We also define the *tent space* $T_{\Omega}^{p}(\mathcal{L})$ is defined as the spaces of functions f so that $\mathcal{A}_{\Omega}^{\infty}(f) \in L^p(\partial\mathcal{L}, ds)$, where p is finite and with norm $||f||_{T_{\Omega}^{p}} = ||\mathcal{A}_{\Omega}^{\infty}(f)||_{L^p(\partial\mathcal{L})}$.

Suppose $\Omega = \{\Omega_{\tilde{x}}\}_{\tilde{x}\in\partial\mathcal{L}}$, where F is any subset of $\partial\mathcal{L}$. We define the *tent* over F, with respect to Ω as

$$\widehat{F_{\Omega}} = \mathcal{L} \setminus \cup_{\tilde{x}\notin F}\Omega_{\tilde{x}}.$$

We also set $\Omega_{\tilde{x}}(t) = \{\tilde{y} \in \partial\mathcal{L} : (y,t) \in \Omega_{\tilde{x}}\}$. For a measure μ in $\mathcal{L}$, we say μ is an (Ω, β)-Carleson measure ($\beta \geq 1$) and write $\mu \in V_{\Omega}^{\beta}$ if

$$||\mu||_{V_{\Omega}^{\beta}} = \sup_{Q\subset\partial\mathcal{L}} \frac{|\mu|(\widehat{Q_{\Omega}})}{|Q|^{\beta}} < \infty,$$

where the supremum is taken over all cubes $Q \subset \partial\mathcal{L}$. Some relevant definitions and results are given in [1],[2] and [4]. Throughout this paper, points on $\partial\mathcal{L}$ will be denoted by $\tilde{x}, \tilde{y}, ...,$ etc.

3. Duality and atomic decomposition of $T_{\Omega}^{p}(\mathcal{L})(0 < p \leq 1)$ space

LEMMA 1. *suppose $F \subset \partial\mathcal{L}$, $\Omega = \{\Omega_{\tilde{x}}\}_{\tilde{x}\in\partial\mathcal{L}}$ are as above. Then*

(i) $A_{\Omega}^{\infty}(\chi_{\widehat{F_{\Omega}}})(\tilde{x}) \leq \chi_F(\tilde{x})$ *for all* $\tilde{x} \in \partial\mathcal{L}$.

(ii) $A_{\Omega}^{\infty}(\chi_{\widehat{F_{\Omega}}})(\tilde{x}) = \chi_F(\tilde{x})$ *if and only if* $\Omega_{\tilde{x}} \cap \widehat{F_{\Omega}} \neq \emptyset$ *for all* $\tilde{x} \in F$.

(iii) *If Ω is a symmetric family (that is, if $\tilde{x} \in \Omega_{\tilde{y}}(t)$ then $\tilde{y} \in \Omega_{\tilde{x}}(t)$), we have that*

$$\widehat{F_{\Omega}} = \{(\tilde{y}, t) \in \mathcal{L} : \Omega_{\tilde{y}}(t) \subset F\}.$$

Proof. (i) Observe that

$$\chi_{\widehat{F_\Omega}}(\tilde{y},t) = \begin{cases} 1 & \text{if } (\tilde{y},t) \notin \Omega_{\tilde{z}} \text{ for all } \tilde{z} \notin F \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Suppose $\tilde{x} \notin F$. If $(\tilde{y},t) \in \Omega_{\tilde{x}}$, then we have $\chi_{\widehat{F_\Omega}}(\tilde{y},t) = 0$ (by (3.1)) and this shows(i).

(ii) $A^\infty_\Omega(\chi_{\widehat{F_\Omega}})(\tilde{x}) = \chi_F(\tilde{x})$ if and only if for all $\tilde{x} \in F$, $A^\infty_\Omega(\chi_{\widehat{F_\Omega}})(\tilde{x}) = 1$ if and only if there exists $(\tilde{y},t) \in \Omega_{\tilde{x}}$ such that $(y,t) \in \widehat{F_\Omega}$ if and only if $\Omega_{\tilde{x}} \cap \widehat{F_\Omega} \neq \emptyset$

(iii) That $(y,t) \in \widehat{F_\Omega}$ means that $y \notin \Omega_{\tilde{x}}(t)$, for all $\tilde{x} \notin F$, which,by symmetry, is equivalent to saying that for all $\tilde{x} \notin F, \tilde{x} \notin \Omega_{\tilde{y}}(t)$; that is, $\Omega_{\tilde{y}}(t) \subset F$.

Let $\mathcal{L}$ be a Lipschitz domain in R^{n+1}_+. A measurable function $a : \mathcal{L} \longrightarrow R$ is a T^p_Ω-*atom* if there exists a ball $Q \subset \partial\mathcal{L}$ such that *supp* $a \subset \widehat{Q_\Omega}$, and $||a||_\infty \leq |Q|^{-\frac{1}{p}}$. We restrict ourselves to the case $n = 1$, but a similar proof also works in any other dimension.

THEOREM 1. *If $\Omega = \{\Omega_{\tilde{x}}\}_{\tilde{x}\in\partial\mathcal{L}}$ is a symmetric family of sets, such that $\Omega_{\tilde{x}}(t)$ is an open for all $(\tilde{x},t) \in \mathcal{L} \subset R^2_+$, then , for $0 < p \leq 1$, $f \in T^p_\Omega$ if and only if*

$$f \equiv \sum_j \lambda_j a_j, \tag{3.2}$$

where a_j is a T^p_Ω-atom and $\sum_j |\lambda_j|^p < \infty$.

Moreover, $||f||_{T^p_\Omega} \approx inf\{(\sum_j |\lambda_j|^p)^{\frac{1}{p}}\}$, where the infimum is taken over all sequences satisfying (3.2).

Proof. We first show that $|| \quad ||_{T^p_\Omega}$ is always a p-norm, for $0 < p \leq 1$ and hence if $f \equiv \sum_j \lambda_j a_j$, then $||f||^p_{T^p_\Omega} \leq \sum_j |\lambda_j|^p ||a_j||^p_{T^p_\Omega}$. But by the previous Lemma:

$$\begin{aligned} ||a_j||^p_{T^p_\Omega} &= \int_{\partial\mathcal{L}} (A^\infty_\Omega(a_j)(\tilde{x}))^p ds \\ &\leq \int_{\partial\mathcal{L}} ||a_j||^p_\infty (A^\infty_\Omega(\chi_{\widehat{Q_{j,\Omega}}})(\tilde{x}))^p ds \\ &\leq ||a_j||^p_\infty \int_{\partial\mathcal{L}} \chi_{Q_j}(\tilde{x}) ds \leq 1 \end{aligned}$$

hence, $||f||^p_{T^p_\Omega} \leq \sum_j |\lambda_j|^p$. For the converse we need the following observation: If $f \in T^p_\Omega$ and $\lambda > 0$ then $\{\tilde{x} : A^\infty_\Omega(f)(x) > \lambda\}$ is an open set. In fact, if $A^\infty_\Omega(f)(\tilde{x}) > \lambda$, then there exists a point $(z,t) \in \Omega_{\tilde{x}}$ so that $|f(z,t)| > \lambda$. By hypothesis, $\tilde{x} \in \Omega_{\tilde{z}}(t)$ and there exists $\epsilon > 0$ such that if $\tilde{y} \in B(\tilde{x},\epsilon)$ then $\tilde{y} \in \Omega_{\tilde{z}}(t)$. Again,by symmetry, $(z,t) \in \Omega_{\tilde{y}}$ and so $A^\infty_\Omega(f) > \lambda$ if $\tilde{y} \in B(\tilde{x},\epsilon)$. Set now $M_k = \{\tilde{x} \in \partial\mathcal{L} : A^\infty_\Omega(f) > 2^k\}$ and write $M_k = \cup_{j\in Z} B^k_j$ by Whitney decomposition([3],[5]). Since $f \in T^p_\Omega$, B^k_j is bounded for all $j,k \in Z$. Set $a_{j,k} \equiv \lambda^{-1}_{j,k} f(\chi_{\widehat{B^k_{j,\Omega}}} - \sum_{B^{k+1}_l \subset B^k_j} \chi_{\widehat{B^{k+1}_{l,\Omega}}})$, where $\lambda_{j,k} = 2^{k+1} s(B^k_j)^{\frac{1}{p}}$. It is clear that $supp\ a_{j,k} \subset \widehat{B^k_{j,\Omega}}$ and

$$\sum_{j,k} |\lambda_{j,k}|^p = \sum_k 2^{p(k+1)} s(M_k)$$
$$\leq C||f||^p_{T^p_\Omega} < \infty$$

and so it remains to show that $f \equiv \sum_{j,k} \lambda_{j,k} a_{j,k}$ and $||a_{j,k}||_\infty \leq s(B^k_j)^{-\frac{1}{p}}$. Let $(x,t) \in \widehat{B^k_{j,\Omega}}$ and suppose $|f(x,t)| > 2^{k+1}$. If $\tilde{y} \in \Omega_{\tilde{x}}(t)$, then $(x,t) \in \Omega_{\tilde{y}}$ and hence $\tilde{y} \in M_{k+1}$. Therefore $\Omega_{\tilde{x}}(t) \subset M_{k+1}$ and there exists a unique $l \in Z$ so that $\Omega_{\tilde{x}}(t) \subset B^{k+1}_l$. Since $\Omega_{\tilde{x}}(t) \subset B^k_j$ then $B^{k+1}_l \subset B^k_j$. Thus,

$$\chi_{\widehat{B^k_{j,\Omega}}}(x,t) - \sum_{B^{k+1}_r \subset B^k_j} \chi_{\widehat{B^{k+1}_{r,\Omega}}}(x,t) = 0.$$

Therefore, for all $(x,t) \in \widehat{B^k_{j,\Omega}}$, $|a_{j,k}(x,t)| \leq 2^{-(k+1)} s(B^k_j)^{-\frac{1}{p}} 2^{k+1} = s(B^k_j)^{-\frac{1}{p}}$.

Finally, if $(x,t) \in \mathcal{L}$ and $2^l < |f(x,t)| < 2^{l+1}$ then $\Omega_{\tilde{x}}(t) \subset M_l$. Let $K \in Z$ be the greast integer satisfying $\Omega_{\tilde{x}}(t) \subset M_k$ (since $A^\infty_\Omega(f)(\tilde{x}) < \infty\, a,e\quad \tilde{x} \in \partial\mathcal{L}$) . Let $s \in Z$ so that $\Omega_{\tilde{x}}(t) \subset B^K_s$. We want to show that if

$$g_{j,k}(x,t) = \chi_{\widehat{B^k_{j,\Omega}}}(x,t) - \sum_{B^{k+1}_r \subset B^k_j} \chi_{\widehat{B^{k+1}_{r,\Omega}}}(x,t)$$

then $\sum_{j,k} g_{j,k}(x,t) = 1$. If $\Omega_{\tilde{x}}(t) \subset B^k_j$, then $k \leq K$. Suppose that $k < K$ and $(x,t) \in \widehat{B^k_{j,\Omega}}$, then $B^K_s \subset B^{k+1}_r \subset B^k_j$ for some $r \in Z$

and hence $g_{j,k}(x,t) = 0$. If $(x,t) \in \widehat{B^K_{j,\Omega}}$ then clearly $j = s$ and $g_{j,K}(x,t) = 1$.

THEOREM 2. *Suppose Ω is a symmetric family and $\{\tilde{x} \in \partial\mathcal{L} : \Omega_{\tilde{x}} \cap K \neq \emptyset\}$ is finite measure. where K is compact set in $\mathcal{L} \subset R^2_+$. For $0 < p \leq 1$, the dual space of T^p_Ω is the space of $V_\Omega^{\frac{1}{p}}$ -Carleson measure. more presisely, the pairing*

$$(f, d\mu) \longrightarrow \int_{\mathcal{L}} f(x,t)d\mu(x,t)$$

with f ranging over functions which are in T^p_Ω and are continuous in $\mathcal{L}$ and $d\mu$ over Carleson measures, relizes the duality of T^p_Ω with Carleson measures $V_\Omega^{\frac{1}{p}}$. That is, $(T^p_\Omega)^* = V_\Omega^{\frac{1}{p}}$

Proof. Let $f \in T^p_\Omega$ and $\mu \in V_\Omega^{\frac{1}{p}}$, and write $f \equiv \sum_j \lambda_j a_j$ as Theorem 1. Then,

$$\begin{aligned}
\mid \int_{\mathcal{L}} f(x,t)d\mu(x,t) \mid &\leq \sum_j |\lambda_j| \int_{\widehat{B_{j,\Omega}}} |a(x,t)|d\mu(x,t) \\
&\leq \sum_j |\lambda_j| ||a_j||_\infty |\mu|(\widehat{B_{j,\Omega}}) \\
&\leq \sum_j |\lambda_j| s(B_j)^{-\frac{1}{p}} ||\mu||_{V_\Omega^{\frac{1}{p}}} s(B_j)^{\frac{1}{p}} \\
&\leq (\sum_j |\lambda_j|^p)^{\frac{1}{p}} ||\mu||_{V_\Omega^{\frac{1}{p}}}.
\end{aligned}$$

Conversely, a bounded functional on $T^p_\Omega(\mathcal{L})$ gives bounded linear functional on $\mathcal{C}(K)$ which is the space of continuous function on compact set K, K ranges over the compact subset of $\mathcal{L}$. This induces a measure $d\mu$ on $\mathcal{L}$. To show $d\mu$ is a Carleson measure. Write $d\mu = \eta d|\mu|$ and put $f(x,t) = \overline{\eta}\chi_{(\widehat{Q})}$. Let $\{f_n\}$ be a sequence of continuous functions with compact support which converges to $f = \overline{\eta}\chi_{\widehat{Q}}$ in the sence of $T^p_\Omega(\mathcal{L})$-norm convergence. Since $A^\infty_\Omega(f) = \chi_Q$, by the continuity of the liner functional we get

$$\begin{aligned}
\int_{\widehat{Q_\Omega}} d|\mu| \leq \mid \int_{\widehat{Q_\Omega}} \overline{\eta} d\mu| &= \mid \int_{\widehat{Q_\Omega}} f d\mu| \\
&\leq ||A^\infty_\Omega(f)||^{\frac{1}{p}}_{L^p(\partial\mathcal{L},ds)} = Cs(Q)^{\frac{1}{p}}.
\end{aligned}$$

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Department of Mathematics
Kyungpook National University
Taegu, 702-701, Korea