Kangweon–Kyungki Math. Jour. **3** (1995), No. 2, pp. 211–222

THE CHOQUET Φ–INTEGRAL WITH RESPECT TO NON–MONOTONIC FUZZY Φ–MEASURES

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1. Introduction

In T. Murofushi , M. Sugeno and M.Machida ([1],[2],[3]), L.M. De Campos and M. Jorge Bolaños[4], they discussed some properties of the Choquet integral with respect to non-monotonic fuzzy measures. Furthermore, T.Muroshi, M.Sugeno and M.Machida[5] investigated the Choquet integral with respect to non-monotonic fuzzy measures of bounded variation. In addition, L.Jang and J. Kwon [6] studied some properties of non-monotonic fuzzy measures of Φ-bounded variation.

In this paper, we introduce the concept of non-monotonic fuzzy measures of Φ-bounded variation, where $\Phi = \{\phi_n\}$ is a sequence of increasing convex functions, defined on the nonnegative real numbers, such that $\phi_n(0) = 0$ and $\phi_n(x) > 0$ for $x > 0$ and $n = 1, 2, \cdots$. We say that Φ is a Φ^*-sequence if and only if $\phi_{n+1}(x) \le \phi_n(x)$ for all n and x , and a Φ-sequence if in addition $\sum_n \phi_n(x)$ diverges for $x > 0$. These definitions were introduced in M. Schramn [7]. Throughout this paper we assume that $(X, \mathcal{F})$ is a measurable space.

In section 2, we will define a non-monotonic fuzzy Φ-measure and the Choquet Φ-integral. In section 3, we discuss some properties of the Choquet Φ-integral with respect to non-monotonic fuzzy Φ-measures.

Received June 28, 1995.

2. Definitions and Preliminaries.

The fuzzy measure and fuzzy integral, defined on a classical σ-algebra, were introduced by M. Sugeno [8].

DEFINITION 2.1 ([5],[6]). *A fuzzy measure on* $(X, \mathcal{F})$ *is a real-valued set function* $\lambda : \mathcal{F} \to R^+$ *satisfying*
(i) $\lambda(\phi) = 0$
(ii) $\lambda(A) \leq \lambda(B)$ *whenever* $A \subset B$ *and* $A, B \in \mathcal{F}$ *where* $R^+ = [0, \infty)$, *the set of nonnegative real numbers.*

Note that in this paper, we do not deal with fuzzy measures λ for which $\lambda(X) = \infty$. In T. Murofuschi, M.Sugeno and M. Machida [5], they discussed non-monotonic fuzzy measures, which are set functions without monotonicity.

DEFINITION 2.2 ([5],[6]). *A non-monotonic fuzzy measure on* $(X, \mathcal{F})$ *is a real-valued set function* $\lambda : \mathcal{F} \to R^+$ *satisfying* $\lambda(\phi) = 0$.

Let $\Phi = \{\phi_n\}$ be either a Φ^*-sequence or a Φ-sequence. In [6], the total Φ-variation $\Phi V(\mu)$ of μ on X is defined by

$$\Phi V(\mu) = \sup \left\{ \sum_{i=1}^{n} \phi_i |\mu(A_i) - \mu(A_{i-1})| \middle| \phi = A_0 \subset \cdots \subset A_n = X, \right\},$$

where $\{A_i\}_{i=0}^n \subset \mathcal{F}$. A real-valued set function μ is said to be of Φ-bounded variation if and only if $\Phi V(\mu) < \infty$. We remark that if $\Phi = \{\phi_n\}$ is the uniformly equicontinuous on $\mathbf{R}$, that is, there is a positive constant M, independent of $n \in \mathbf{N}$ and $x, y \in \mathbf{R}$, such that

$$|\phi_n(x) - \phi_n(y)| \leq M|x - y|$$

then, the Proposition 2.5([6]) implies that a monotonic fuzzy measure λ is of Φ-bounded variation. We denote the set of monotonic fuzzy measures on $(X,\mathcal{F})$ by $FM(X,\mathcal{F})$ and the set of non-monotonic fuzzy measures of Φ-bounded variation on $(X,\mathcal{F})$ by $\Phi BV(X,\mathcal{F})$. Then, the Theorem 2.9([6]) implies that $\Phi BV(X,\mathcal{F})$ is a real Banach space with $\|\cdot\|_\Phi$, where

$$\|\mu\|_\Phi = \inf\{k > 0 \ : \ \Phi V\left(\frac{\mu}{k}\right) \leq 1\}$$

for every $\mu \in \Phi BV(X,\mathcal{F})$.

DEFINITION 2.3 ([6]). *For every $\mu \in \Phi BV(X,\mathcal{F})$, we define*

$$|\mu|_\Phi(A) = \sup\left\{\sum_{i=1}^{n}\phi_i(|\mu(A_i)-\mu(A_{i-1})|)\middle|\phi = A_0 \subset \cdots \subset A_n = A\right\}$$

$$\mu_\Phi^+(A) = \sup\left\{\sum_{i=1}^{n}\phi_i([\mu(A_i)-\mu(A_{i-1})]^+)\middle|\phi = A_0 \subset \cdots \subset A_n = A\right\}$$

$$\mu_\Phi^-(A) = \sup\left\{\sum_{i=1}^{n}\phi_i([\mu(A_i)-\mu(A_{i-1})]^-)\middle|\phi = A_0 \subset \cdots \subset A_n = A\right\}$$

where $\{A_i\}_{i=0}^n \subset \mathcal{F}$, $[r]^+ = \max\{r,0\}$ and $[r]^- = \max\{-r,0\}$. We call $|\mu|_\Phi$, μ_Φ^+, and μ_Φ^-, the total Φ-variation, positive total Φ-variation, negative total Φ-variation of μ, respectively.

DEFINITION 2.4 ([6]). *Let Φ be either Φ^*-sequence or Φ-sequence and let μ be a non-monotonic fuzzy measure on $(X,\mathcal{F})$ of Φ-bounded variation. Then μ_Φ is defined by*

$$\mu_\Phi(A) = \mu_\Phi^+(A) - \mu_\Phi^-(A), \ \textit{for every } A \in \mathcal{F}$$

In this case, we say that μ_Φ is a non-monotonic fuzzy Φ-measure on $(X,\mathcal{F})$.

3. Characterizations of Choquet Φ-integrals.

T.Murofushi, M.Sugeno and M.Machida[5] introduced the Choquet integral with respect to non-monotonic fuzzy measures. And also, they discussed the Choquet integral with respect to non-monotonic fuzzy measures of bounded variation. In this section, we define the Choquet Φ-integral with respect to non-monotonic fuzzy Φ-measures and investigate some characterizations of the Choquet Φ-integral.

DEFINITION 3.1. *The Choquet Φ-integral of a measurable function $f : X \to \mathbf{R}$ with respect to a non-monotonic fuzzy Φ-measure μ_Φ is defined by*

$$(C)\int f d\mu_\Phi = \int_{-\infty}^{\infty} \mu_{\Phi f}(r)dr$$

whenever the integral in the right-hand side exists. Here, $\mu_{\Phi f}(r)$ is defined by

$$\mu_{\Phi f}(r) = \begin{cases} \mu_\Phi(\{x \mid f(x) > r\}), & \text{for } r \geq 0 \\ \mu_\Phi(\{x \mid f(x) > r\}) - \mu_\Phi(X), & \text{for } r < 0. \end{cases}$$

We note that $\mu^+_{\Phi f}$ *and* $\mu^-_{\Phi f}$ are defined by

$$\mu^+_{\Phi f}(r) = \begin{cases} \mu^+_\Phi(\{x \mid f(x) > r\}), & \text{for } r \geq 0 \\ \mu^+_\Phi(\{x \mid f(x) > r\}) - \mu^+_\Phi(X), & \text{for } r < 0. \end{cases}$$

and

$$\mu^-_{\Phi f}(r) = \begin{cases} \mu^-_\Phi(\{x \mid f(x) > r\}), & \text{for } r \geq 0 \\ \mu^-_\Phi(\{x \mid f(x) > r\}) - \mu^-_\Phi(X), & \text{for } r < 0. \end{cases}$$

respectively. Since $\mu_\Phi(A) = \mu^+_\Phi(A) - \mu^-_\Phi(A)$ for each $A \in \mathcal{F}$, it is easy to show that

$$(C)\int f d\mu_\Phi = (C)\int f d\mu^+_\Phi - (C)\int f d\mu^-_\Phi.$$

A measurable function f is called Φ-integrable on X if the Choquet Φ-integral of f exists and its value is finite.

PROPOSITION 3.2. *Let $\phi_n(x) = x$ for all n and let $f : X \to \mathbf{R}$ be a measurable function. Then we have*

$$(C)\int f d\mu_\Phi = (C)\int f d\mu$$

where $(C)\int f d\mu$ is the Choquet integral of f with respect to a non-monotonic fuzzy measure μ.

PROOF. Since $\phi_n(x) = x$ for all n,

$$\begin{aligned}
\mu_{\Phi f}(r) &= \begin{cases} \mu_\Phi(\{x \mid f(x) > r\}), & \text{for } r \geq 0 \\ \mu_\Phi(\{x \mid f(x) > r\}) - \mu_\Phi(X), & \text{for } r < 0. \end{cases} \\
&= \begin{cases} \mu(\{x \mid f(x) > r\}), & \text{for } r \geq 0 \\ \mu(\{x \mid f(x) > r\}) - \mu(X), & \text{for } r < 0. \end{cases} \\
&= \mu_f(r) \qquad \text{for each } r \in \mathbf{R}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
(C)\int f d\mu_\Phi &= \int_{-\infty}^{\infty} \mu_{\Phi f}(r) dr \\
&= \int_{-\infty}^{\infty} \mu_f(r) dr = (C)\int f d\mu.
\end{aligned}$$

From Proposition 3.2, we note that if $\phi_n(x) = x$ for all n, then a measurable function f is Φ-integrable if and only if it is integrable.

PROPOSITION 3.3. *If μ_Φ, ν_Φ are non-monotonic fuzzy Φ-measures, if a and b are real numbers, and if f is a measurable function, then*

$$(C)\int f d(a\mu_\Phi + b\nu_\Phi) = a(C)\int f d\mu_\Phi + b(C)\int f d\nu_\Phi$$

PROOF. If $r \geq 0$, then we have

$$\begin{aligned}
(a\mu_\Phi + b\nu_\Phi)_f(r) &= (a\mu_\Phi + b\nu_\Phi)(\{x \mid f(x) > r\}) \\
&= a\mu_\Phi(\{x \mid f(x) > r\}) + b\nu_\Phi(\{x \mid f(x) > r\}) \\
&= (a\mu_\Phi)_f(r) + (b\nu_\Phi)_f(r)
\end{aligned}$$

If $r < 0$, then we have

$$\begin{aligned}
&(a\mu_\Phi + b\nu_\Phi)_f(r)\\
&= (a\mu_\Phi + b\nu_\Phi)(\{x \mid f(x) > r\}) - (a\mu_\Phi + b\nu_\Phi)(X)\\
&= a\left[\mu_\Phi(\{x \mid f(x) > r\}) - \mu_\Phi(X)\right] + b\left[\nu_\Phi(\{x \mid f(x) > r\}) - \nu_\Phi(X)\right]\\
&= (a\mu_\Phi)_f(r) + (b\nu_\Phi)_f(r)
\end{aligned}$$

Hence, for all $r \in \mathbf{R}$,

$$(a\mu_\Phi + b\nu_\Phi)_f(r) = a\mu_{\Phi f}(r) + b\nu_{\Phi f}(r).$$

Therefore, we obtain

$$\begin{aligned}
&(C)\int f d(a\mu_\Phi + b\nu_\Phi)\\
&= (C)\int_{-\infty}^{\infty} (a\mu_\Phi + b\nu_\Phi)_f(r)dr\\
&= (C)\int_{-\infty}^{\infty} a\mu_{\Phi f}(r) + b\nu_{\Phi f}(r)dr\\
&= a(C)\int_{-\infty}^{\infty} \mu_{\Phi f}(r)dr + b\int_{-\infty}^{\infty} \nu_{\Phi f}(r)dr\\
&= a(C)\int f d\mu_\Phi + b(C)\int f d\nu_\Phi.
\end{aligned}$$

Now the following are some properties of the Choquet Φ-integral.

PROPOSITION 3.4. *For every $A \in \mathcal{F}$, $(C)\int 1_A d\mu_\Phi = \mu_\Phi(A)$.*

PROOF. Assume that $r \geq 0$. If $r \geq 1$, then we have

$$\mu_\Phi(\{x \mid 1_A(x) > r\}) = \mu_\Phi(\phi) = 0$$

If $0 \leq r < 1$, then we have

$$\mu_\Phi(\{x \mid 1_A(x) > r\}) = \mu_\Phi(A).$$

Assume that $r < 0$. Then we have

$$\mu_\Phi(\{x \mid 1_A(x) > r\}) = \mu_\Phi(X).$$

And hence, we have

$$\mu_\Phi(\{x \mid 1_A(x) > r\}) - \mu_\Phi(X) = 0, \quad \text{for each } r < 0.$$

Therefore, we obtain

$$\begin{aligned}(C)\int 1_A d\mu_\Phi &= \int_{-\infty}^{\infty} \mu_{\Phi 1_A}(r) d\mu \\ &= \int_0^1 \mu_{\Phi 1_A}(r) dr = \int_0^1 \mu_\Phi(A) dr = \mu_\Phi(A).\end{aligned}$$

Let us consider the Choquet Φ-integral of a nonnegative simple function. Every nonnegative simple function f on X can be represented by

$$f = \sum_{i=1}^{n} a_i 1_{D_i} \tag{3.1}$$

where $0 \leq a_1 < \cdots < a_n < \infty$, $D_i \cap D_j = \phi$ for $i \neq j$ and $X = \cup_{i=1}^n D_i$.

PROPOSITION 3.5. *Let f be a nonnegative simple function as in (3.1). Then*

$$(C)\int f d\mu_\Phi = \sum_{i=1}^{n} (a_i - a_{i-1}) \mu_\Phi(A_i)$$

where $A_i = \cup_{k=i}^n D_k$ for $i = 1, 2, \cdots, n$ and $a_0 = 0$.

PROOF. If $r < 0 = a_0$,

$$\mu_{\Phi f}(r) = \mu_\Phi(\{x \mid f(x) > r\}) - \mu_\Phi(X) = \mu_\Phi(X) - \mu_\Phi(X) = 0.$$

If $a_{i-1} < r < a_i$ for each $i = 1, 2, \cdots, n$, then

$$\mu_{\Phi f}(r) = \mu_\Phi(\{x \mid f(x) > r\}) = \mu_\Phi(\cup_{k=i}^n D_k) = \mu_\Phi(A_i),$$

since $A_i = \cup_{k=i}^n D_k$ for $i = 1, 2, \cdots, n$. Therefore, we obtain

$$\begin{aligned}
(C)\int f d\mu_\Phi &= \int_{-\infty}^{\infty} \mu_{\Phi f}(r)dr \\
&= \int_0^{\infty} \mu_{\Phi f}(r)dr \\
&= \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \mu_{\Phi f}(r)dr + \int_{a_n}^{\infty} \mu_{\Phi f}(r)dr \\
&= \sum_{i=1}^{\infty} \int_{a_{i-1}}^{a_i} \mu_\Phi(A_i)dr \\
&= \sum_{i=1}^{n} (a_i - a_{i-1})\mu_\Phi(A_i).
\end{aligned}$$

We denote by $B(X, \mathcal{F})$ the set of bounded measurable functions on $(X, \mathcal{F})$. Then $B(X, \mathcal{F})$ is a real Banach space with respect to the norm defined by

$$\|f\| = \sup_{x \in X} |f(x)| .$$

And also, we denote by $B^+(X, \mathcal{F})$ the set of nonnegative bounded measurable functions on $(X, \mathcal{F})$.

PROPOSITION 3.6. *If $f \in B^+(X, \mathcal{F})$ and a is any nonnegative real number, then*

$$(C)\int af d\mu_\Phi = a(C)\int f d\mu_\Phi.$$

PROOF. If $a = 0$, then $af(x) = 0$ for each $x \in X$. So, we have

$$\mu_{\Phi(af)}(r) = \begin{cases} \mu_\Phi(\{x \mid af(x) > r\}), & \text{for } r \geq 0 \\ \mu_\Phi(\{x \mid af(x) > r\}) - \mu_\Phi(X), & \text{for } r < 0. \end{cases}$$
$$= \begin{cases} \mu(\phi), & \text{for } r \geq 0 \\ \mu(X) - \mu(X), & \text{for } r < 0. \end{cases}$$
$$= 0 \quad \text{for each } r \in \mathbf{R}.$$

Hence

$$(C)\int afd\mu_\Phi = \int_{-\infty}^{\infty} \mu_{\Phi(af)}(r)dr = 0 = a(C)\int fd\mu_\Phi.$$

If $a > 0$, we have

$$\mu_{\Phi(af)}(r) = \begin{cases} \mu_\Phi(\{x \mid af(x) > r\}), & \text{for } r \geq 0 \\ \mu_\Phi(\{x \mid af(x) > r\}) - \mu_\Phi(X), & \text{for } r < 0. \end{cases}$$
$$= \begin{cases} \mu_\Phi(\{x \mid f(x) > \frac{r}{a}\}), & \text{for } r \geq 0 \\ \mu_\Phi(\{x \mid f(x) > \frac{r}{a}\}) - \mu_\Phi(X), & \text{for } r < 0. \end{cases}$$
$$= \mu_{\Phi f}\left(\frac{r}{a}\right)$$

Hence

$$\begin{aligned} (C)\int afd\mu_\Phi &= \int_{-\infty}^{\infty} \mu_{\Phi(af)}(r)dr \\ &= a\int_{-\infty}^{\infty} \mu_{\Phi f}(\frac{r}{a})d\frac{r}{a} \\ &= a\int_{-\infty}^{\infty} \mu_{\Phi f}(s)ds \\ &= a(C)\int fd\mu_\Phi \end{aligned}$$

where $s = \frac{r}{a}$.

PROPOSITION 3.7. *If $f \in B^+(X, \mathcal{F})$, then*

$$(C)\int f d\mu_\Phi = \int_0^\infty \mu_{\Phi f}(r)dr.$$

PROOF. For $r < 0$,

$$\begin{aligned}\mu_{\Phi f}(r) &= \mu_\Phi\{x \in X \mid f(x) > r\} - \mu_\Phi(X) \\ &= \mu_\Phi(X) - \mu_\Phi(X) = 0\end{aligned}$$

Hence

$$\begin{aligned}(C)\int f d\mu_\Phi &= \int_{-\infty}^\infty \mu_{\Phi f}(r)dr \\ &= \int_0^\infty \mu_{\Phi f}(r)dr + \int_{-\infty}^0 \mu_{\Phi f}(r)dr \\ &= \int_0^\infty \mu_{\Phi f}(r)dr\end{aligned}$$

PROPOSITION 3.8. *If $f, g \in B^+(X, \mathcal{F})$ and $f(x) \leq g(x)$ for every $x \in X$, then*

$$(C)\int f d\mu_\Phi^+ \leq (C)\int g d\mu_\Phi^+.$$

PROOF. For each $r \geq 0$, we put

$$A^r = \{x \in X \mid f(x) > r\} \text{ and } B^r = \{x \in X \mid g(x) > r\}.$$

Hence we obtain

$$\begin{aligned}&\mu_{\Phi f}^+(r) \\ &= \mu_\Phi^+\{x \in X \mid f(x) > r\} \\ &= \mu_\Phi^+(A^r) \\ &= \sup\left\{\sum_{i=1}^n \phi_i([\mu(A_i^r) - \mu(A_{i-1}^r)]^+)\middle| \phi = A_0^r \subset \cdots \subset A_n^r = A^r\right\} \\ &\leq \sup\left\{\sum_{i=1}^n \phi_i([\mu(A_i^r) - \mu(A_{i-1}^r)]^+)\middle| \phi = A_0^r \subset \cdots \subset A_n^r = B^r\right\} \\ &= \mu_{\Phi g}^+,\end{aligned}$$

where $\{A_i^r\}_{i=0}^n \subset \mathcal{F}$. Therefore, we have

$$(C)\int f d\mu_\Phi^+ \leq (C)\int g\mu_\Phi^+.$$

We remark that

(i) by the similarity of the proof of the proposition 3.8, it is easy to show that

$$(C)\int f d\mu_\Phi^- \leq (C)\int g\mu_\Phi^-.$$

under the same hypotheses ;

(ii) in general, it is not true that $(C)\int f d\mu_\Phi \leq (C)\int g\mu_\Phi$, whenever $f, g \in B^+(X, \mathcal{F})$ and $f(x) \leq g(x)$ for every $x \in X$;

(iii) from the proposition 3.6, the Choquet Φ-integral functional on $B^+(X, \mathcal{F})$ satisfies positively homogeneous. Here, the definition of positively homogeneous was introduced by T. Murofushi, M. sugeno and M. Machida [5].

References

[1]. T. Murofushi, M. Sugeno and M. Machida, *An interpretation of fuzzy measures and the Choquet integral as integral with respect to a fuzzy measure*, Fuzzy Sets and Systems **29** (1989), 201-227.

[2]. T. Murofushi, M. Sugeno and M. Machida, *Some quantities represented by the Choquet integral*, Fuzzy Sets and Systems **56** (1993), 229-235.

[3]. T. Murofushi, M. Sugeno and M. Machida, *Fuzzy t-conorm integral generalization of Sugeno integral and Choquet integral*, Fuzzy Sets and Systems **42** (1991), 57-71.

[4]. L.M.de Campos and M.Jorge Bolaños, *Characterization and comparison of Sugeno and Choquet integrals*, Fuzzy Sets and Systems **52** (1992), 61-67.

[5]. T. Murofushi, M. Sugeno and M.Machida, *Non-monotonic fuzzy measures and the Choquet integral*, Fuzzy Sets and Systems **64** (1994), 73-86.

[6]. L. Jang and J. Kwon, *On non-monotonic fuzzy measures of Φ-bounded variation*, Fuzzy Sets and Systems, submitted.

[7]. M. Schramn, *Function of Φ-bounded variation and Riemann-Stieltjes integration*, Trans. A.M.S. **287** (1985), 49-63.

[8]. M. Sugeno, *Theory of fuzzy integrals and its applications*, Thesis, Tokyo Institute of Technology (1974).

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