REMARKS ON SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

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1. Introduction.

A normal almost contact structure such that its fundamental 2-form and contact form are both closed is called cosymplectic structure. A Riemannian manifold with a cosymplectic structure is said to be cosymplectic manifold [1, 2, 3]. In this paper, we are to construct almost complex structure F on a product manifold of an anti-invariant submanifold of cosymplectic manifolds and Euclidean spaces. Moreover, we study the necessary and sufficient conditions for F is integrable. In the last, we introduce the almost Hermitian structure on the product manifold. Unless otherwise stated we use in the present paper the system of indices as follows:

$$A, B, C, D, \cdots \ : 1, 2, \cdot, 2m + 2$$
$$a, b, c, d, \cdots \ : 1, 2, \cdots, n$$
$$x, y, z, w, \cdots \ : n + 1, \cdots, n + p = 2m + 1$$
$$* \ : 2m + 2.$$

2. Anti-invariant submanifolds of cosymplectic manifolds.

Let M be a $(2m+1)$-dimensional cosymplectic manifold with structure (ϕ, ξ, η, G), that is, a manifold M which admits a 1-form η, a vector field ξ, a metric tensor G satisfying

$$(2.1) \quad \begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \\ &G(\xi, X) = \eta(X), \quad \eta(\xi) = 1, \\ &G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y) \end{aligned}$$

(2.2) $d\phi = 0, \quad d\xi = 0$

for any vector fields X, Y on M. The fundamental 2-form Φ is defined by

(2.3) $\Phi(X, Y) = G(\phi X, Y).$

It can be shown in [1] that the cosymplectic structure is characterized by

(2.4) $\nabla_X \Phi = 0 \quad and \quad \nabla_X \eta = 0,$

where ∇ is the connection of G.

Let N be an n-dimensional anti–invariant submanifold of M, that is,

$$\phi(T_x(N)) \subset T_x(M)^\perp$$

for each point x in N, where $T_x(N)$ and $T_x(M)^\perp$ denote the tangent and normal space of N at x, respectively. The induced metrics on the tangent and normal space are respectively defined by

$$g_{cb} = G(B_c, B_b)$$
$$g_{xy} = G(C_x, C_y).$$

We denote the inverse components of $g = (g_{cb})$ and $\bar{g} = (g_{xy})$ by $g^{-1} = (g^{cb})$ and $\bar{g}^{-1} = (g^{xy})$ respectively. Then we get [4,5,6],

(2.5)
$$\phi_x^{\ y}\phi_y^{\ z} = -\delta_x^{\ z} + \xi_x\xi^z + f_x^{\ a}f_a^{\ z},$$
$$\phi_x^{\ b}\xi^x = 0,$$
$$\phi_a^{\ x}\eta^a = \phi_z^{\ x}\xi^z,$$
$$\eta^a\eta_a + \xi_x\xi^x = 1,$$
$$\phi_c^{\ x}\phi_x^{\ y} = -\eta_c\xi^y,$$
$$\phi_c^{\ x}\phi_x^{\ b} = \delta_c^{\ b} - \eta_c\eta^b,$$

where we have put

$$G(\phi B_c, C_x) = -\phi_{cx},$$
$$G(\phi C_x, B_a) = \phi_{xa},$$
$$G(\phi C_x, C_y) = \phi_{xy},$$
$$G(\xi, B_a) = \eta_a,$$
$$G(\xi, C_x) = \xi_x$$

for B_c and C_x the local basis of the tangent space and normal space of N, and

$$\phi_{xy}g^{yz} = \phi_x{}^z,$$
$$\phi_{xa}g^{ab} = \phi_x{}^b,$$
$$\eta_a g^{ab} = \eta^b,$$
$$\xi_x g^{xy} = \xi^y.$$

We easily see that

$$\phi_{xy} = -\phi_{yx} \text{ and } \phi_{cx} = \phi_{xc}.$$

Let ∇_d be the induced Riemannian connection on N, then we get [4]

$$\nabla_d \phi_x{}^y = h_{da}{}^x \phi_a{}^y - h_{da}{}^y f_x{}^a,$$
$$\nabla_d \phi_c{}^y = -h_{dc}{}^x f_x{}^y,$$
(2.6)
$$\nabla_d \eta^a = h_{da}{}^x \xi^x,$$
$$\nabla_d \xi_x = -h_{da}{}^x \eta^a,$$
$$h_{bc}{}^x \phi_x{}^a = \phi_c{}^x h_{ba}{}^x.$$

3. An almost complex structure on $N \times R^p \times R^1$.

In this section, we consider the product manifold $N \times R^p \times R^1$, R^p being a p-dimensional Euclidean space. Denote $N \times R^p \times R^1$ by $\tilde{M}$ and define on $\tilde{M}$ a tensor field F of type (1.1) with local components $F_B{}^A$ given by

(3.1)
$$(F_B{}^A) = \begin{pmatrix} 0 & \phi_b{}^x & \eta_b \\ -\phi_y{}^a & \phi_y{}^x & -\xi_y \\ -\eta^a & \xi^x & 0 \end{pmatrix}$$

in $\{\tilde{M}, x^A\}$, $\{N, x^a\}$ being a coordinate neighborhood of N and $(x^{n+1}, \cdots, x^{n+p} = x^{2m+1})$ being a Cartesian coordinate in R^p and $x^* = x^{2m+2}$ natural coordinate in R^1. Then, taking account of (2.5), we can see that $F^2 = -I$ holds on $\tilde{M}$. Thus we have

PROPOSITION 3.1. *If N is an anti–invariant submanifold of cosymplectic manifolds, then $\tilde{M} = N \times R^p \times R^1$ is an almost complex manifold.*

The Nijenhuis tensor $[F, F]_{CB}^{A}$ of the almost complex structure F has local components

$$[F, F]_{CB}^{A} = F_C^E \partial_E F_B^A - F_B^E \partial_E F_C^A - (\partial_C F_B^E - \partial_B F_C^E) F_E^A$$

on $\tilde{M}$. We denote $[F, F]_{CB}^{A}$ by N_{CB}^{A}. Then, using (3.1), the components of N_{CB}^{A} are given by as follows;

$$(3.2) \qquad N_{yz}^{x} = \phi_x^e \partial_e \phi_y^z - \phi_y^e \partial_e \phi_x^z,$$

$$(3.3) \qquad N_{yz}^{a} = \phi_y^e \partial_e \phi_z^a - \phi_z^e \partial_e \phi_y^a,$$

$$(3.4) \qquad N_{yz}^{*} = \phi_y^e \partial_e \xi_x - \phi_x^e \partial_e \xi_y$$

and so forth.

It is well known that the Nijenhuis tensor of an almost complex structure F satisfies the condition [6]

$$(3.5) \qquad N_{CE}^{A} F_B^E + N_{CB}^{E} F_E^A = 0.$$

Substituting (3.1) into (3.5), we have

(3.6-1)
$$N_{cy}^{\ a}\phi_x^{\ y} - N_{ce}^{\ a}\phi_x^{\ e} - N_{cx}^{\ y}\phi_y^{\ a} - N_{c*}^{\ a}\xi_x + N_{cx}^{\ *}\eta^a = 0,$$

(3.6-2)
$$N_{cx}^{\ e}\phi_e^{\ y} - N_{ce}^{\ y}\phi_x^{\ e} + N_{cz}^{\ y}\phi_x^{\ z} + N_{cx}^{\ z}\phi_z^{\ y} - N_{c*}^{\ y}\xi_x + N_{cx}^{\ *}\xi^y = 0,$$

(3.6-3)
$$N_{cx}^{\ e}\eta_e - N_{ce}^{\ *}\phi_x^{\ e} + N_{cy}^{\ *}\phi_x^{\ y} - N_{cx}^{\ y}\xi_y - N_{c*}^{\ *}\xi_x = 0,$$

(3.6-4)
$$N_{xz}^{\ a}\phi_y^{\ z} - N_{xe}^{\ a}\phi_y^{\ e} - N_{xy}^{\ z}\phi_z^{\ a} - N_{x*}^{\ a}\xi_y - N_{xy}^{\ *}\eta^a = 0,$$

(3.6-5)
$$N_{xz}^{\ e}\phi_e^{\ y} - N_{xe}^{\ y}\phi_z^{\ e} + N_{xw}^{\ y}\phi_z^{\ w} + N_{xz}^{\ w}\phi_w^{\ y} - N_{x*}^{\ y}\xi_z + N_{xz}^{\ *}\xi^y = 0,$$

(3.6-6)
$$N_{xy}^{\ e}\eta_e - N_{xe}^{\ *}\phi_y^{\ e} + N_{xz}^{\ *}\phi_y^{\ z} - N_{xy}^{\ z}\xi_z - N_{x*}^{\ x}\xi_y = 0,$$

(3.6-7)
$$N_{*y}^{\ a}\phi_x^{\ y} - N_{*e}^{\ a}\phi_x^{\ e} + N_{*x}^{\ y}\phi_y^{\ a} - N_{*x}^{\ *}\eta^a = 0,$$

(3.6-8)
$$N_{*x}^{\ e}\phi_e^{\ y} - N_{*e}^{\ y}\phi_x^{\ e} + N_{*z}^{\ y}\phi_x^{\ z} + N_{*x}^{\ z}\phi_z^{\ y} + N_{*x}^{\ *}\xi_y = 0,$$

(3.6-9)
$$N_{*x}^{\ e}\eta_e - N_{*e}^{\ *}\phi_x^{\ e} + N_{*y}^{\ *}\phi_x^{\ y} - N_{*x}^{\ y}\xi_y = 0.$$

Now we assume that $N_{yz}^{\ x}, N_{yz}^{\ a}, N_{yz}^{\ *}$ are vanish identically and the function $\lambda^2 = g_{xy}\xi^x\xi^y$ does not vanish almost everywhere. The the equation (3.6-4) becomes

(3.7)
$$N_{xe}^{\ a}\phi_y^{\ e} + N_{x*}^{\ a}\xi_y = 0.$$

Transvecting (3.7) with ξ^y, then we get $N_{x*}^{\ a} = 0$, so that (3.7) is reduced to

(3.8)
$$N_{xe}^{\ a}\phi_y^{\ e}0.$$

Transvecting $\phi_c^{\ y}$ and η^c successively, we get $N_{zc}^{\ a} = 0$. By the same method, we obtain $N_{z*}^{\ x} = 0$ and $N_{zc}^{\ x} = 0$ from (3.6-5), $N_{z*}^{\ *} = 0$ and

$N_{ze}^* = 0$ from (3.6-6) respectively. Then the equations (3.6-7), (3.6-8) and (3.6-9) are reduced to

(3.9) $N_{*e}^a \phi_y^{\ e} = 0,$

(3.10) $N_{*e}^x \phi_y^{\ e} = 0,$

(3.11) $N_{*e}^* \phi_y^{\ e} = 0.$

Transvecting $\phi_c^{\ y}$ and using (2.5), we get $N_{*e}^a = 0$, $N_{*e}^x = 0$ and $N_{*e}^* = 0$. Henceforth the equations (3.6-1), (3.6-2) and (3.6-3) become

(3.12) $N_{ce}^a \phi_y^{\ e} = 0,$

(3.13) $N_{ce}^x \phi_y^{\ e} = 0,$

(3.14) $N_{ce}^* \phi_y^{\ e} = 0.$

By the same method of the above statements, we obtain $N_{ce}^a = 0$, $N_{ce}^x = 0$ and $N_{ce}^* = 0$. Thus we have

THEOREM 3.2. *Let N be an anti–invariant submanifold of cosymplectic manifolds and let the function λ^2 does not vanish almost everywhere. If the three components N_{yz}^x, N_{yz}^a and N_{yz}^* are identically vanish on N, then the other components are all vanish.*

It is well known that [3,6]

THEOREM 3.3. *In order that an almost complex structure F be integrable, it is necessary and sufficient that the components of N are vanish.*

Considering Theorems 3.2 and 3.3, we get

THEOREM 3.4. *Let N be an anti–invariant submanifold of cosymplectic manifolds and let the function λ^2 does not vanish almost everywhere. Then the almost complex structure F is integrable if and only if the components N_{yz}^x, N_{yz}^a and N_{yz}^* are identically vanish on N.*

By use of the induced Riemannian connection ∇_d on N, we can calculate the components of N_{yz}^x, N_{yz}^a and N_{yz}^* as follows:

(3.15) $N_{yz}^x = (h_e^{\ d}{}_y \phi_z^{\ e} - h_e^{\ d}{}_z \phi_y^{\ e}) \phi_d^{\ x},$

(3.16) $N_{yx}^a = h_d^{\ az}(\phi_{zy}\phi_x^{\ d} - \phi_{zx}\phi_y^{\ d}),$

(3.17) $N_{yx}^* = (h_{dey}\phi_x^{\ e} - h_{dex}\phi_y^{\ e})\eta^d,$

with the aid of (2.6). Assume that $N_{yz}^{\ x} = 0$ and apply $\phi_x^{\ b}$ and η_b successively to (3.15), we get

$$(3.18) \qquad\qquad (h_e^{\ d}{}_y \phi_z^{\ e} - h_e^{\ d}{}_z \phi_y^{\ e}) \eta_d = 0,$$

that is, $N_{yz}^{\ *} = 0$. Therefore we can state the Theorem 3.4 as follows.

THEOREM 3.5. *Let N be an anti–invariant submanifold of cosymplectic manifolds and let the function λ^2 does not vanish almost everywhere. Then the almost complex structure F is integrable if and only if the components $N_{yz}^{\ x}$, $N_{yz}^{\ a}$ are identically vanish on N.*

4. Almost Hermitian structure on $\tilde{M}$.

Let N be an anti–invariant submanifold of cosymplectic manifolds. If we consider a Riemannian metric H on $\tilde{M}$ with components

$$(4.1) \qquad H = (h_{CB}) = \begin{pmatrix} g_{cb} & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

then we can see that

$$(4.2) \qquad\qquad H_{CB} F_D^{\ C} F_E^{\ B} = H_{DE},$$

that is, (F, H) defines an almost Hermitian structure on $\tilde{M}$. Thus we can state

THEOREM 4.1. *The almost complex manifold $(\tilde{M}, F)$ becomes an almost Hermitian manifold with a Hermite metric H.*

We denote the christoffel symbol by $\{\widetilde{B\ ^C A}\}$ and $\{b\ ^c a\}$ formed by $H = (h_{CB})$ and $g = (g_{cb})$ respectively. Then, using (4.1), we find

$$\{\widetilde{b\ ^c a}\} = \{b\ ^c a\}$$

and the other are vanish.

Let $\tilde{\nabla}$ be a covariant differentiation with respect to $\{\widetilde{B\ ^C A}\}$ formed by H. Then the covariant derivatives of F are given by

$$(4.3) \qquad\qquad \tilde{\nabla}_c F_b{}^x = \nabla_c \phi_b{}^x,$$

$$(4.4) \qquad\qquad \tilde{\nabla}_c F_b{}^* = \nabla_c \eta_b,$$

$$(4.5) \qquad\qquad \tilde{\nabla}_c F_y{}^x = \nabla_c \phi_y{}^x,$$

$$(4.6) \qquad\qquad \tilde{\nabla}_c F_y{}^* = -\nabla_c \xi_y$$

and the others are all vanish. Hence we have

PROPOSITION 4.2. *The almost Hermitian manifold $\tilde{M}$ to be a Kaehlerian space with (F, H) if and only if all of $\phi_y{}^x$, $\phi_b{}^x$, ξ^x and η_b are covariantly constant on N.*

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