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THE OPERATOR AND PROBABILITY ON A HYPERGROUP

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1. Notations and Preliminaries

Let K be a locally compact hypergroup introduced by *convo* in [10]. For a Borel function f on K, f^- is defined through involution and complex conjugation at the same time, and the mapping $x \to x^-$ is called an involution of K. The Dirac measure in a point $x \in K$ will be abbreviated by ε_x. Let $\mathcal{B}(\mathfrak{H})$ be the Banach $*$-algebra of all bounded linear operators on a Hilbert space $\mathfrak{H}$, and I the identity operator. We shall say that U is a representation of K on $\mathfrak{H}$ if the following four conditions are satisfied :

(1) The mapping $\mu \to U_\mu$ is a $*$-homomorphism from $\mathfrak{M}(K)$ into $\mathcal{B}(\mathfrak{H})$ where $\mathfrak{M}(K)$ is the Banach algebra under the convolution of all finite regular Borel measures on K.
(2) If $\mu \in \mathfrak{M}(K)$ then $\|U_\mu\| \leq \|\mu\|$.
(3) $U_{\varepsilon_x} = I$.
(4) If $a, b \in \mathfrak{H}$ then the mapping $\mu \to< U_\mu a, b >$ is continuous on $\mathfrak{M}^+(K)$ with respect to the cone topology.

We shall write U_x for U_{ε_x} for $x \in K$. By condition (4), if $a, b \in \mathfrak{H}$ then the mapping $x \to< U_x a, b >$ is bounded and continuous, and

$$< U_\mu a, b >= \int_K < U_x a, b > \mu(dx)$$

for all $\mu \in \mathfrak{M}^1(K)$, where $\mathfrak{M}^1(K)$ is the space of probability measures on K furnished with the weak topology.

Let χ be continuous and not indentically zero, complex-valued function on K with the property $\chi(x * y) = \chi(x)\chi(y)$ for all $x, y \in K$. We denote the set of all such functions by $\mathfrak{X}(K)$ and the set of all χ in $\mathfrak{X}(K)$ where are bounded will be denoted by $\mathfrak{X}_b(K)$. In particular,

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$\chi \in \mathfrak{X}_b(K)$ with $|\chi(x)| = 1$, $x \in K$, is called a *character* of K. For χ in $\mathfrak{X}_b(K)$, let F_χ be defined on $\mathfrak{M}(K)$ by

$$F_\chi(\mu) = \int_K \overline{\chi} d\mu ,$$

and $\widehat{K}$ be the set of all χ in $\mathfrak{X}_b(K)$ such that

$$\chi(x^-) = \overline{\chi(x)}$$

for x in K. Moreover a function χ on K will be called a *normalized character* if $\chi = \frac{1}{n}T$ where T is the trace function of an irreducible unitary representation of K on an n-dimensional Hilbert space. Recall that a function is a normalized character if and only if χ is continuous and

$$\chi(x)\chi(y) = \int_K \chi(t^- xty)\omega(dt)$$

for all $x, y \in K$, where ω is the normalized Haar measure. For $x \in K$ the conjugacy class of x is defined by $x^G = \{t^- xt \,|\, t \in K\}$ and if H is the hypergroup of conjugacy classes for K, the characters of H are in one-to-one correspondence with the normalized character χ_ρ of representation ρ in $\mathcal{R}$. We denote by $\mathfrak{F}_c$, $\mathfrak{F}_u$, and $\mathfrak{F}_{co}$, respectively, the algebras of continuous, unformly continuous, and continuous with compact support ; by $\mathfrak{F}_\gamma(\chi)$, $\mathfrak{F}_z(K)$, respectively, the subalgebra of $\mathfrak{F}_u$ consisting of representative functions associated with a continuous irreducible unitary representation (we will denote the totality of all such representations by $\mathcal{R}$), subalgebra of $\mathfrak{F}_c$ consisting of the central functions, *i.e.*, $f(x) = f(yxy^-)$ for all x, y in K. Also $\int_{K/Z} d\dot{x}$ denotes the normalized Haar integral on K/Z, and $\int_K dx$ is a left invariant Haar integral on K ; ω_K denotes the Haar measure on K.

Finally, Dunkl [3] defines the center $Z = Z(K)$ of a hypergroup K as the set of all x in K such that $\mathrm{supp}(\varepsilon_x * \varepsilon_y)$ is a singleton for each $y \in K$. Jewett [10] defines the maximal subgroup $G(K)$ of K as the set of all x in K such that $\mathrm{supp}(\varepsilon_x * \varepsilon_{x^-}) = \{e\}$. If K/Z is compact, then a locally compact hypergroup K is called *central* or Z-hypergroup. Such hypergroups arise natually as double coset spaces of compact subgroups of Z-groups.

If f is a Borel function on K and $x, y \in K$ then we define $f(x * y) = f^y(x) = \int_K f(z) d\varepsilon_x * \varepsilon_y(z)$ if this integral exists, and we will denote it by $f(xy)$. For any $[0, \infty]$−valued function f, g and left invariant Haar measure m, the convolution $f * g$ of f and g is defined on K by $(f * g)(x) = \int_K f(x * y) g(y^{-1}) m(dy)$.

2. Representation

PROPOSITION 2.1. *Let K be a Z−hypergroup and let ρ be a continuous irreducible unitary representation of K on the complex Hilbert space $\mathcal{H}$. Then*

(1) *For each choice of u, v, u', v' in $\mathcal{H}$ the function defined by*

$$x \to < \rho_x u, v > \overline{< \rho_x u', v' >}$$

(where $x \in K$ and $-$ denotes complex conjugation) is constant on cosets of Z and is in $\mathfrak{F}_c(K/Z)$.

(2) *To ρ corresponds a positive real number $c_\rho = \int_{K/Z} | < \rho_x u, v > |^2 d\dot{x}$, where u and v are any vector in $\mathcal{H}$ of norm 1, and*

$$\int_{K/Z} < \rho_x u, v > \overline{< \rho_x u', v' >} d\dot{x} = c_\rho < u, u' > \overline{< v, v' >}$$

for all u, v, u', v' in $\mathcal{H}$

Proof. Let $z \in Z$. Since ρ is a irreducible unitary representation of K, $\rho_x \rho_z = \rho_z \rho_x$ for each x in K and $\rho_z = \lambda(z) I$ where $\lambda(z) \in C, |\lambda(z)|^2 = 1$ and I is the identity on $\mathcal{H}$. Since ρ is continuous, and

$$\begin{aligned} < \rho_z \rho_x u, v > \overline{< \rho_z \rho_x u', v' >} &= \lambda(z) < \rho_x u, v > \lambda(z)^{-} \overline{< \rho_x u', v' >} \\ &= < \rho_x u, v > \overline{< \rho_x u', v' >}, \end{aligned}$$

the function in (1) is a constant on cosets of Z, and is in $\mathfrak{F}_c(K/Z)$. Now consider the function

$$\phi(u, v, u', v') = \int_{K/Z} < \rho_x u, v > \overline{< \rho_x u', v' >} d\dot{x}\,.$$

For fixed u, u', this function is linear in v' and conjugate linear in v. Moreover, $|\phi(u,v,u',v')| \le \|u\|\|v\|\|u'\|\|v'\|$ because of the unitarity of ρ. Hence we can represent $\phi(u,v,u',v')$ by a bounded operator. Specifically, there exists a family $B_{u,u'}$ of bounded operators on $\mathcal{H}$ satisfying $< B_{u,u'}v', v >= \phi(u,v,u',v')$. Then

$$
\begin{aligned}
< B_{u,u'}\rho_y v', v > &= \int_{K/Z} < \rho_{yx}u, v > \overline{< \rho_{yx}u', v' >} d\dot{x} \\
&= \int_{K/Z} < \rho_{yx}u, v > \overline{< \rho_{x}u', v' >} d\dot{x} \\
&\quad \text{by the invariance of the Haar integral} \\
&= \int_{K/Z} < \rho_{x}u, \rho_y^* v > \overline{< \rho_{x}u', v' >} d\dot{x} \\
&=< B_{u,u'}v', \rho_y^* v > \\
&=< \rho_y B_{u,u'}v', v >
\end{aligned}
$$

for all $y \in K$ and $u, v, u', v' \in \mathcal{H}$. Hence $B_{u,u'}\rho_y = \rho_y B_{u,u'}$ for all $y \in K$ and $u, u' \in \mathcal{H}$, so that by Schur's lemma $B_{u,u'} = \lambda(u,u')I, \lambda(u,u') \in \mathbb{C}$. Hence $\phi(u,v,u',v') =< B_{u,u'}v', v >= \lambda(u,u') < v', v >$. On the other hand, since ρ is unitary,

$$
< \rho_x u, v > \overline{< \rho_x u', v' >} =< u, \rho_{x^{-1}} v > \overline{< u', \rho_{x^{-1}} v' >}
$$

and hence

$$
< B_{u,u'}v', v >= \int_{K/Z} < u, \rho_{x^{-1}} v > \overline{< u', \rho_{x^{-1}} v' >} d\dot{x} \, .
$$

Moreover since K/Z is unimodular

$$
\begin{aligned}
\int_{K/Z} < u, \rho_{x^{-1}} v > \overline{< u', \rho_{x^{-1}} v' >} d\dot{x} &= \int_{K/Z} < u, \rho_x v > \overline{< u', \rho_x v' >} d\dot{x} \\
&=< B_{v,v'}u', u > \\
&= \lambda(v,v') < u, u' > \, .
\end{aligned}
$$

We conclude that $\lambda(u,u') < v', v >= \lambda(v,v') < u, u' >$. So $\lambda(v,v)$ is constant on $\|v\| = 1$ and does not depend on v. If we denote $\lambda(v,v)$

by c_ρ, then $\lambda(u,u') = c_\rho < u, u' >$ and

$$\int_{K/Z} < \rho_x u, v > \overline{< \rho_x u', v' >} d\dot{x} = c_\rho < u, u' >< v', v >$$
$$= c_\rho < u, u' > \overline{< v, v' >},$$

and if $u = u'$, $v = v'$ and $\|u\| = \|v\| = 1$, we obtain $c_\rho = \int_{K/Z} | < \rho_x u, v > |^2 d\dot{x}$. ■

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup in $\mathfrak{M}^1(K)$. If we define $\psi_x(u) =< \rho_x u, u >$, then ψ_x will be a corresponding quadratic form of $(\mu_t)_{t\geq 0}$. Now we recall that $\mathcal{P}(K)$ is the cone of all bounded continuous positive definite functions on K. Let $\mathcal{P}^1(K)$ be the convex set of $\phi \in \mathcal{P}(K)$ such that $\phi(e) = \|\phi\|_\infty = 1$, and let, further, π_ϕ be the corresponding representation to $\phi \in \mathcal{P}^1(K)$. Then we have a relation $|\phi(yx) - \phi(x)| \leq 2(1 - \mathrm{Re}\phi(y^-))$ for $\phi \in \mathcal{P}^1(K)$ and $x, y \in K$, and the canonical mapping $\phi \to \pi_\phi$ from $\mathrm{ex}\mathcal{P}^1(K)$ onto $\widehat{K}$ is open and continuous, where $\mathrm{ex}\mathcal{P}^1(K)$ is the set of extreme points of $\mathcal{P}^1(K)$ endowed with the topology of uniform convergence on compact subset of K (pp94-95, [7]).

THEOREM 2.2. *Let K be a $Z-$hypergroup. If ρ is a continuous irreducible unitary representation of K, then ρ is finite dimensional, and the dimension function $\rho \to d_\rho$ is continuous, locally bounded on $\widehat{K}$.*

Proof. Let $\{v_i | i \in I\}$ be a maximal orthogonal family in $\mathcal{H}$. For any integer n, we have

$$\sum_{i=1}^{n} | < \rho_x v_i, v_1 > |^2 \leq \sum_{i\in I} | < \rho_x v_i, v_1 > |^2,$$

thus by proposition 2.1,

$$\int_{K/Z} \sum_{i=1}^{n} | < \rho_x v_i, v_1 > |^2 d\dot{x} = \sum_{i=1}^{n} \int_{K/Z} | < \rho_x v_i, v_1 > |^2 d\dot{x}$$
$$= nc_\rho \leq \int_{K/Z} \sum_{i\in I} | < \rho_x v_i, v_1 > |^2 d\dot{x}.$$

Since ρ is unitary, $\{\rho_x v_i\}$, for every x, is also a maximal orthogonal family in $\mathcal{H}$. Moreover we have

$$\|v_1\|^2 = \sum_{i \in I} | < \rho_x v_i, v_1 > |^2 = 1$$

by Parseval's equation. Hence $nc_\rho \leq 1$ and since c_ρ is a positive real number, n is a bounded integer. So, ρ has a finite dimension. Let us denote by d_ρ the dimension of ρ. Since K/Z is compact it follows that $C = \cup_{x \in K}\{x\} * \{x^-\}$ is a compact subset of K(2.5D,2.5F,3.2C, [10]), and we can set $K = CZ$ (proposition 4.3, [16]). Let $\phi_0 \in \mathrm{ex}\mathcal{P}^1(K)$ and $\epsilon > 0$ be given. Then the set

$$W = \{\pi_\phi | \phi \in \mathrm{ex}\mathcal{P}^1(K), |\phi(x) - \phi_0(x)| < \epsilon \quad \text{for all } x \in C\}$$

is an open neighborhood of π_ϕ on $\widehat{K}$. Since

$$|\phi(x) - \phi_0(x)| = | < \pi_\phi(x)u, u > - < \pi_{\phi_0}(x)u_0, u_0 > |$$

for some $u_0 \in \mathcal{H}_{\rho_0}$, $\|u_0\| = 1$, and $u \in \mathcal{H}_{\pi_\phi}$, $\|u_\phi\| = 1$, we get

$$\begin{aligned}
&\left| \int_{K/Z} \left(\|\pi_{\phi_0}(x)u_0\|^2 - \|\pi_\phi(x)u\|^2\right) d\dot{x} \right| \\
&\leq \int_{K/Z} | < \pi_{\phi_0}(x * x^{-1})u_0, u_0 > - < \pi_\phi(x * x^{-1})u, u > | d\dot{x} \\
&\leq \int_{K/Z} \left(\int_K | < \pi_{\phi_0}(y)u_0, u_0 > - < \pi_\phi(y)u, u > | d\varepsilon_x * \varepsilon_{x^{-1}}(y) \right) d\dot{x} \\
&\leq \int_{K/Z} \int_K \epsilon d\varepsilon_x * \varepsilon_{x^{-1}}(y) d\dot{x} = \epsilon
\end{aligned}$$

for all $\pi_\phi \in W$. Thus $\rho \to \int_{K/Z} \|\rho_x u\|^2 d\dot{x}$ is continuous. On the other hand if $\{v_1, \cdots, v_{d_\rho}\}$ is an orthogonal basis of $\mathcal{H}$, then we have, as the above, $c_\rho d_\rho \leq 1$. Since $\rho \to c_\rho$ is continuous $\rho \to d_\rho$ is locally bounded and continuous. ■

From the theorem 2.2, if $\{v_1, \cdots, v_{d_\rho}\}$ is an orthogonal basis of $\mathcal{H}$, then for each $i = 1, 2, \cdots, d_\rho$, we have

$$d_\rho c_\rho = \sum_{i=1}^{d_\rho} \int_{K/Z} | < \rho_x v_i, v_1 > |^2 d\dot{x} = \int_{K/Z} \sum_{i=1}^{d_\rho} | < \rho_x v_i, v_1 > |^2 d\dot{x} .$$

By the Parseval's equation, we obtain that $d_\rho c_\rho = 1$. hence we see from this fact and proposition 2.1 that

$$\int_{K/Z} | < \rho_x u, v > |^2 d\dot{x} = d_\rho^{-1} \quad \text{for any } u, v \in \mathcal{H} \text{ and } \|u\| = \|v\| = 1$$

and for each u, v, u', v' in $\mathcal{H}$,

$$\int_{K/Z} < \rho_x u, v > \overline{< \rho_x u', v' >} d\dot{x} = d_\rho^{-1} < u, u' > \overline{< v, v' >}.$$

If $\rho = \pi_\phi \in \widehat{K}$, $\phi \in \mathrm{ex}\mathcal{P}^1(K)$, then $\rho_z = \phi(z)I$ for all $z \in Z$ and $\phi|Z \in \widehat{Z}$ where $\widehat{Z}$ denotes the character group of Z. We therefore have a canonical mapping $r : \widehat{K} \to \widehat{Z}$ defined by $\pi_\phi \to \phi|Z$. This mapping is continuous, open and surjective ([4],[7]). Moreover we can easily see that if $\rho_x = (\rho_{ij}(x))$ is the matrix representation of ρ relative to the above basis, then

$$\int_{K/Z} \rho_{li}(x) \overline{\rho_{jk}(x)} d\dot{x} = d_\rho^{-1} \delta_{ik} \delta_{jl}, \quad \text{for} \quad i, j, k, l = 1, 2, \cdots, d_\rho,$$

on the other words,

$$\int_{K/Z} < \rho_x v_l, v_i > \overline{< \rho_x v_j, v_k >} d\dot{x} = d_\rho^{-1} \delta_{ik} \delta_{jl}$$

and in particular $\int_{K/Z} |\chi_\rho|^2 d\dot{x} = 1$([4],[7]).

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