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FUZZY NEARNESS SPACES

YONG CHAN KIM

1. INTRODUCTION

S.K. Samanta [12] introduced the notion of fuzzy nearness structures. A. Kandil and M.E. El-Shafee [5] investigated some characterizations for FR_0-spaces.

In this paper we will obtain that the category of the topological fuzzy nearness spaces is isomorphic to the category of the FR_0-spaces. In particular, the category of the fuzzy nearness spaces have initial structures and final structures. In fact, by the concepts of initial structures and final structures, we can define subspaces and products and quotients of fuzzy nearness spaces.

Throughout this paper, for general categorical background we refer to J. Admek, H. Herrlich, and G.E. Strecker [1] and for nearness spaces to Gerhard Preuss [11] and C.Y. Kim et al. [7].

2. PRELIMINARIES

We introduce some basic definitions, notations and known results. We will denote by I the unit interval and by I^X the family of all fuzzy subsets of a set X. The constant function whose value is α is denoted by $\tilde{\alpha}$.

Let $A, B \in I^X$. We say A and B are *quasi-coincident* (briefly q.c.), denoted by $A \; q \; B$, iff there exists $x \in X$ such that $A(x) + B(x) > 1$. Otherwise we write $A \; \bar{q} \; B$. $A \subseteq B$ if $A(x) \leq B(x)$, $(A \cup B)(x) = max\{A(x), B(x)\}$, $(A \cap B)(x) = min\{A(x), B(x)\}$, for all $x \in X$.

A *fuzzy point* $x_t, 0 < t \leq 1$, is an element of I^X such that $x_t(x) = t$ and $x_t(y) = 0$ for all $y \in \{x\}^c$. A fuzzy point $x_t \in A$ iff $t \leq A(x)$.

If $f : X \to Y$ is a function, then for any $A \in I^X$ the *image* of A is defined as $f(A)(y) = sup_{x \in f^{-1}(\{y\})} A(x)$ for every $y \in Y$, where the fact $sup \, \emptyset = \emptyset$ is assumed, and if for any $B \in I^Y$ the *preimage* of B is defined as $f^{-1}(B) = B \circ f$.

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If $f : X \to Y$ is a function, then we remember the following properties for direct and inverse image of fuzzy sets:

(i) if $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$,

(ii) $f(f^{-1}(A)) \subseteq A$ with equality if f is surjective,

(iii) $f^{-1}(f(E)) \supseteq E$ with equality if f is injective,

(iv) $f^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{-1}(A_i)$,

(v) $f(\bigcup_{i\in I} E_i) = \bigcup_{i\in I} f(E_i)$.

Let X be a nonempty set. A map $c : I^X \to I^Y$ is said to be an *fuzzy closure operator* on X iff it satisfies the following conditions:

(c1) $c(\emptyset) = \emptyset$,

(c2) $A \subseteq c(A)$ for every $A \in I^X$,

(c3) $c(A \cup B) = c(A) \cup c(B)$ for every $A, B \in I^X$.

The pair (X, c) is a *fuzzy closure space* (briefly FC-space). A FC-space (X, c) is a *topological FC-space* if (c4) $c \circ c(A) = c(A)$ for all $A \in I^X$.

Let (X, c) be a FC-space. Then $\tau_c = \{A \in I^X \mid A^c = c(A^c)\}$ is a fuzzy topology on X, where $A^c = \tilde{1} - A$.
Let $(X, c), (Y, c^*)$ be FC-spaces and $f : X \to Y$. Then f is a *c-map* iff $f(c(A)) \subseteq c^*(f(A))$ for all $A \in I^X$.

We can easily prove the following lemma.

LEMMA 2.1. [5] *Let X be a nonempty set. Then:*

(i) If $A, B \in I^X$, $A\ q\ B$, then $A \cap B \neq \emptyset$.

(ii) For $A, B \in I^X$, $A\ \bar{q}\ B$ iff $A \subseteq B^c$.

(iii) For $A, B \in I^X$, $A \subseteq B$ iff $x_t\ q\ A$ implies $x_t\ q\ B$ iff $x_t\ \bar{q}\ A$ for all $x_t\ \bar{q}\ B$ iff $x_t \in B$ for all $x_t \in A$.

(iv) For $A_i \in I^X$ for all $i \in I$, $x_t\ q\ \bigcup_{i\in I} A_i$ iff there exists $i_0 \in I$ such that $x_t\ q\ A_{i_0}$.

3. Fuzzy nearness spaces

In this section, we study relationships between fuzzy nearness spaces and fuzzy closure spaces.

Let X be a set and $\mathcal{A}, \mathcal{B} \subset I^X$. A family $\mathcal{A}$ of fuzzy subsets is said to be *quasi-coincident* (briefly q.c.) iff there exists $x \in X$ such that $A(x) + B(x) > 1$, for every $A, B \in \mathcal{A}$. We define $\mathcal{A} \vee \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. $\mathcal{A}$ is called a *corefinement* of $\mathcal{B}$, denoted by $\mathcal{A} \prec \mathcal{B}$, iff for every $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $B \subseteq A$.

DEFINITION 3.1. *A subset ξ of $P(I^X)$ is said to be a fuzzy nearness structure (briefly FN-structure) on X if it satisfies the following conditions:*

(FN1) if $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \in \xi$, then $\mathcal{A} \in \xi$,
(FN2) if $\mathcal{A}$ is q.c., then $\mathcal{A} \in \xi$,
(FN3) $\emptyset \neq \xi \neq P(I^X)$,
(FN4) if $\mathcal{A} \vee \mathcal{B} \in \xi$, then $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$,
(FN5) if $\{ cl_\xi A \mid A \in \mathcal{A}\} \in \xi$, then $\mathcal{A} \in \xi$, where

$$cl_\xi A = 1 - sup\{B \in I^X \mid \{B, A\} \notin \xi\}.$$

The pair (X, ξ) is called a fuzzy nearness space (briefly FN-space).

Let ξ and ξ^* be FN-structures for X. We say that ξ is *coarser* than ξ^* and ξ^* is *finer* than ξ if $\xi \subset \xi^*$.

Remark. If we identify a subset A of a set X with the characteristic function χ_A of A on X, then we may consider a nearness structure on X as a fuzzy nearness structure on X. In fact, the notion of fuzzy nearness structures is shown to be a generalization of that of nearness structures.

By the following theorem, we will write cl instead of cl_ξ in (FN5) of Definition 3.1. We can easily prove that cl is a topological fuzzy closure operator on I^X.

THEOREM 3.2. *Let (X, ξ) be a FN-space . Define an operator $cl : I^X \to I^X$ as $x_t\ q\ cl(A)$ iff $\{x_t, A\} \in \xi$. Then:*
(a) $cl = cl_\xi$.
(b) cl is a topological fuzzy closure operator on I^X.

Proof. (a) Let $x_t\ \bar{q}\ cl(A)$ be given. Then $\{x_t, A\} \notin \xi$. Therefore, by the definition cl_ξ, $cl_\xi A(x) \leq 1 - t$, i.e., $x_t\ \bar{q}\ cl_\xi(A)$. By Lemma 2.1, $cl_\xi(A) \subseteq cl(A)$.

We must show that $cl_\xi(A) \supseteq cl(A)$ for all $A \in I^X$. First, we will show that if $\{A, B\} \notin \xi$, then $cl(A)\ \bar{q}\ B$. If $cl(A)\ q\ B$, i.e., $cl(A)(x) + B(x) > 1$ for some $x \in X$, then there exists $t \in I$ such that $cl(A)(x) + t > 1$ and $x_t \in B$. Hence $x_t\ q\ cl(A)$. Since $\{A, B\} \prec \{x_t, A\} \in \xi$, $\{A, B\} \in \xi$. Second, since $\{A, B\} \notin \xi$ implies $cl(A) \subseteq B^c$, we have $cl(A) \subseteq cl_\xi(A)$.

(b) (c1); We have $x_t\ \bar{q}\ \emptyset$ for all x_t. On the other hand, we have $\{x_t, \emptyset\} \notin \xi$, i.e., $x_t\ \bar{q}\ cl(\emptyset)$. For, otherwise, by (FN1), $\mathcal{A} \prec \{x_t, \emptyset\} \in \xi$

for every $\mathcal{A} \in P(I^X)$, i.e., $\xi = P(I^X)$, a contradiction to (FN3). Hence $cl(\emptyset) = \emptyset$.

(c2); Let $x_t \, q \, A$ be given. By (FN2), then $\{x_t, A\} \in \xi$, that is, $x_t \, q \, cl(A)$.

(c3); If $A \subseteq B, \{x_t, B\} \prec \{x_t, A\}$. Therefore $cl(A) \subseteq cl(B)$. Hence $cl(A) \cup cl(B) \subseteq cl(A \cup B)$. Let $x_t \, q \, cl(A \cup B)$ be given. Then $\{x_t, A \cup B\} \in \xi$. By (FN1), $\{x_t, A \cup x_t, B \cup x_t, A \cup B\} \in \xi$. By (FN4), $\{x_t, A\} \in \xi$ *or* $\{x_t, B\} \in \xi$, i.e., $x_t \, q \, cl(A)$ *or* $x_t \, q \, cl(B)$.

(c4); Let $x_t \, q \, cl \circ cl(A)$ be given. Then $cl(x_t) \, q \, cl \circ cl(A)$. By first of the proof (a), $\{cl(x_t), cl(A)\} \in \xi$. By (FN5) and (a), $\{x_t, A\} \in \xi$. Then $x_t \, q \, cl(A)$.

Remark. If (X, ξ) is a nearness space in a sence the above remark, then, by the above theorem, (X, cl) is a closure space where $cl(A) = \{x \in X \mid \{A, \{x\}\} \in \xi\}$.

DEFINITION 3.3. *Let (X, ξ) and (Y, η) be FN-spaces. A map $f : (X, \xi) \to (Y, \eta)$ is called a **fuzzy nearness map** (FN-map) if $f(\mathcal{A}) = \{f(A) \mid A \in \mathcal{A}\} \in \eta$ for each $\mathcal{A} \in \xi$.*

THEOREM 3.4. *Let (X, ξ) and (Y, η) be FN-spaces and let $f : (X, \xi) \to (Y, \eta)$ be a FN-map. Then $f : (X, cl_\xi) \to (Y, cl_\eta)$ is a c-map.*

Proof. We will show that $f(cl_\xi(A)) \subseteq cl_\eta(f(A))$ for all $A \in I^X$.

Let $y_t \, q \, f(cl_\xi(A))$ be given. There exists $x_0 \in X$ such that

$$sup_{x \in f^{-1}(\{y\})} cl_\xi A(x) + t \geq cl_\xi A(x_0) + t > 1,$$

where $f(x_0) = y$. Since $(x_0)_t \, q \, cl_\xi(A)$, i.e., $\{(x_0)_t, A\} \in \xi$, we have $\{f((x_0)_t), f(A)\} \in \eta$, i.e., $f(x_0)_t \, q \, cl_\eta(f(A))$.

DEFINITION 3.5. *A FN-space (X, ξ) is called a **topological FN-space** if $cl_\xi \mathcal{A} = \{cl_\xi(A); A \in \mathcal{A}\}$ is q.c. for each $\mathcal{A} \in \xi$.*

*A topological FC-space (X, c) is called a **FR_0-space** if $x_t \, \bar{q} \, c(y_r)$ implies $y_r \, \bar{q} \, c(x_t)$.*

Remarks. (a) If (X, ξ) is a FN-space, then (X, cl_ξ) is a FR_0-space.

(b) In a sence the characteristic function, we may consider a R_0-space as a FR_0-space.

For categorical viewpoints, we denote

TFN=(topological FN-spaces, FN-maps),

$\mathbf{FR_0}$=(FR_0-spaces, c-maps).

Define a map $F : \mathbf{TFN} \to \mathbf{FR_0}$, $F(X, \xi) = (X, cl_\xi)$ and $F(f) = f$. By Theorem 3.4 and the above remarks, then F is a functor.

LEMMA 3.6. *(X, c) is a FR_0-space iff $x_t \bar{q}\ c(A)$ implies $c(x_t)\ \bar{q}\ c(A)$ for all $A \in I^X$.*

Proof. ($\Rightarrow$) If $x_t\ \bar{q}\ c(A)$, then $t + c(A)(x) \leq 1$ for all $x \in X$. We will show that $c(x_t) \subseteq c(A)^c$. By Lemma 2.1, for every $y_r\ \bar{q}\ c(A)^c$, i.e., $y_r \in c(A)$, we have $c(y_r) \subseteq c(A)$. Hence $x_t\ \bar{q}\ c(y_r)$. Since (X, c) is a FR_0-space, we have $y_r\ \bar{q}\ c(x_t)$. Therefore $c(x_t)\ \bar{q}\ c(A)$.

($\Leftarrow$) Put $c(A) = c(y_r)$. Then $x_t\ \bar{q}\ c(y_r)$ implies $c(x_t)\ \bar{q}\ c(y_r)$. Hence $c(x_t)\ \bar{q}\ y_r$.

THEOREM 3.7. *Let (X, c) be a topological FC-space and let $\xi_c = \{\mathcal{A} \in P(I^X) \mid \{c(A) \mid A \in \mathcal{A}\}$ is q.c. $\}$ and for $A \in I^X$, $x_t\ q\ cl_{\xi_c}(A)$ iff $\{x_t, A\} \in \xi_c$. Then $cl_{\xi_c}A = c(A)$ iff (X, c) is a FR_0-space.*

Proof. ($\Leftarrow$) If $x_t\ q\ c(A)$, then $c(x_t)\ q\ c(A)$. Hence $\{x_t, A\} \in \xi_c$, i.e., $x_t\ q\ cl_{\xi_c}(A)$.

If $x_t\ q\ cl_{\xi_c}(A)$, i.e., $\{x_t, A\} \in \xi_c$, then $c(x_t)\ q\ c(A)$. Since (X, c) is a FR_0-space, by the above lemma, $x_t\ q\ c(A)$.

($\Rightarrow$) Put $c(x_t)\ q\ c(A)$. Then $\{x_t, A\} \in \xi_c$, i.e., $x_t\ q\ cl_{\xi_c}(A)$. Since $cl_{\xi_c}A = c(A)$, we have $x_t\ q\ c(A)$.

Example. Let $X = \{0, 1, 2\}$ be given. Define $\tau = \{A, B, \tilde{0}, \tilde{1}\}$ where $A(0) = 0.2, A(1) = 0.4, A(2) = 0.1, B(0) = 0.8, B(1) = 0.4, B(2) = 0.9$. Since $\bar{0}_{0.5}\ q\ 1_{0.5}$, but $0_{0.5}\ \bar{q}\ \bar{1}_{0.5}$, (X, c) is not a FR_0-space. Suppose $cl_{\xi_c}A(x) < 1$ for some $x \in X$. There exists x_t such that $x_t\ \bar{q}\ cl_{\xi_c}A$. So, $\{x_t, A\} \notin \xi_c$, but $\{c(x_t), c(A)\}$ is q.c., a contradiction. Hence $c(A) = B^c < cl_{\xi_c}A = 1$.

LEMMA 3.8. *If $\mathcal{A} \vee \mathcal{B}$ is q.c., then $\mathcal{A}$ or $\mathcal{B}$ is q.c..*

Proof. See Lemma 1.10 in [12].

THEOREM 3.9. *If (X, c) is a FR_0-space, then $\xi_c = \{\mathcal{A} \in P(I^X) \mid \{c(A) \mid A \in \mathcal{A}\}$ is q.c.$\}$ is a topological FN-structure on X such that $cl_{\xi_c} = c$.*

Proof. The proof of this theorem is similar to the proof of Theorem 1.9 in [12].

LEMMA 3.10. *If (X,ξ) is a topological FN-space, then $\xi_{cl_\xi} = \xi$.*

Proof. If $\mathcal{A} \in \xi_{cl_\xi}$, then $cl_\xi \mathcal{A} = \{cl_\xi(A) \mid A \in \mathcal{A}\}$ is q.c.. Hence $cl_\xi \mathcal{A} \in \xi$. By (FN5), $\mathcal{A} \in \xi$.

If $\mathcal{A} \in \xi$, then $cl_\xi \mathcal{A}$ is q.c., because (X,ξ) is a topological FN-space. Hence $\mathcal{A} \in \xi_{cl_\xi}$

THEOREM 3.11. *Let $(X,\xi),(Y,\eta)$ be topological FN-spaces. A map $f:(X,\xi)\to(Y,\eta)$ be a FN-map iff $f:(X,cl_\xi)\to(Y,cl_\eta)$ is a c-map.*

Proof. ($\Rightarrow$) By Theorem 3.4, it is trivial.

($\Leftarrow$) If $\mathcal{A} \in \xi$, then $cl_\xi(\mathcal{A})$ is q.c.. Since $f(cl_\xi(A)) \subseteq cl_\eta(f(A))$, $\{cl_\eta(f(A)) \mid A \in \mathcal{A}\}$ is q.c. By (FN2), $\{cl_\eta(f(A)) \mid A \in \mathcal{A}\} \in \eta$. By (FN5), $f(\mathcal{A}) = \{f(A) \mid A \in \mathcal{A}\} \in \eta$.

In [7] and [11], it was introduced that the category of the topological nearness spaces is isomorphic to the category of the R_0-spaces.

We will generalize it as the following theorem.

THEOREM 3.12. *(a) If a map $G : \mathbf{FR_0} \to \mathbf{TFN}$ is defined by $G(X,c) = (X,\xi_c)$ and $G(f) = f$, then G is a functor.*

(b) $\mathbf{TFN}$ and $\mathbf{FR_0}$ are isomorphic, i.e., $F\circ G = 1$ and $G \circ F = 1$.

Proof. (a) By Theorem 3.9, (X,ξ_c) is a topological FN-space. Since (X,c) is a FR_0-space, we have $cl_{\xi_c} = c$. By Theorem 3.11, $G(f) = f$ is a FN-map.

(b) Since $F\circ G(X,c) = (X, cl_{\xi_c})$, by Theorem 3.9, we have $F\circ G = 1$. Since $G\circ F(X,\xi) = (X,\xi_{cl_\xi})$, by Lemma 3.10, we have $G\circ F = 1$.

4. INITIAL AND FINAL FUZZY NEARNESS STRUCTURES

Now, we will prove the existences of initial and final fuzzy nearness structures.

Notations; Let $\xi \subseteq P(I^X)$ and $\eta \subseteq P(I^Y)$ be given. If $f : X \to Y$ is a map, then we will write $f(\xi) = \{\mathcal{B} \subseteq I^Y \mid f^{-1}(\mathcal{B}) \in \xi\}$ where $f^{-1}(\mathcal{B}) = \{f^{-1}(B) \mid B \in \mathcal{B}\}$ and $f^{-1}(\eta) = \{\mathcal{A} \subseteq I^X \mid f(\mathcal{A}) \in \eta\}$.

THEOREM 4.1. *(Existence of Initial Structures). If X is a set, $((X_i,\xi_i))_{i\in I}$ is a family in FN-space, and $f_i : X \to X_i$ is a map for each $i \in I$, then:*

(a) $\xi = \bigcap\{f_i^{-1}(\xi_i) \mid i \in I\}$ is the coarsest fuzzy nearness structure on X with respect to which f_i is a FN-map for each $i \in I$.

(b) A map $f : (Z, \zeta) \to (X, \xi)$ is a FN-map if and only if

$$f_i \circ f : (Z, \zeta) \to (X_i, \xi_i), \quad i \in I$$

is a FN-map.

Proof. (a) (FN1); If $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \in \xi$, then $\mathcal{B} \in f_i^{-1}(\xi_i)$ for all $i \in I$. Hence $f_i(\mathcal{B}) \in \xi_i$ and $f_i(\mathcal{A}) \prec f_i(\mathcal{B})$ for all $i \in I$. Thus $f_i(\mathcal{A}) \in \xi_i$ for all $i \in I$. Therefore $\mathcal{A} \in \xi$.

(FN2); If $\mathcal{A}$ is q.c., there exists $x \in X, y = f_i(x)$ such that

$$f_i(A_1)(y) + f_i(A_2)(y) \geq A_1(x) + A_2(x) > 1$$

for any $A_1, A_2 \in \mathcal{A}$. Thus $f_i(\mathcal{A}) \in \xi_i$ for all $i \in I$. Therefore $\mathcal{A} \in \xi$.

(FN3); Since $\mathcal{A} = \{\tilde{1}, \tilde{\frac{1}{2}}\}$ is q.c. and $f_i^{-1}(\xi_i) \neq P(I^X)$, we have $P(I^X) \neq \xi \neq \emptyset$.

(FN4); If $\mathcal{A} \vee \mathcal{B} \in \xi$, then $f_i(\mathcal{A} \vee \mathcal{B}) \in \xi_i$ for all $i \in I$. Since $f_i(A \cup B) = f_i(A) \cup f_i(B)$, $f_i(\mathcal{A}) \vee f_i(\mathcal{B}) \in \xi_i$ for all $i \in I$. By (FN4), $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$.

(FN5); If $\{\ cl_\xi A; A \in \mathcal{A}\} \in \xi$, then $f_i(cl_\xi \mathcal{A}) \in \xi_i$. We will show that $f_i(cl_\xi A) \subseteq cl_{\xi_i}(f_i(A))$. Let $(y_i)_t\ q\ f_i(cl_\xi(A))$ be given. There exists $x_i \in X$ such that

$$sup_{x \in f_i^{-1}(\{y_i\})} cl_\xi A(x) + t \geq cl_\xi A(x_i) + t > 1$$

where $f_i(x_i) = y_i$. Since $(x_i)_t\ q\ cl_\xi(A)$, i.e., $\{(x_i)_t, A\} \in \xi$, we have $\{f_i((x_i)_t), f_i(A)\} \in \eta_i$ for all $i \in I$, i.e., $(y_i)_t\ q\ cl_\eta(f(A))$. Therefore, we obtain $cl_{\xi_i}(f_i(\mathcal{A})) \prec f_i(cl_\xi \mathcal{A}) \in \xi_i$, $\mathcal{A} \in \xi$.

(b) ($\Rightarrow$) It is trivial.

($\Leftarrow$) For every $\mathcal{A} \in \zeta$, we have $f_i \circ f(\mathcal{A}) \in \xi_i$ for all $i \in I$. Hence $f(\mathcal{A}) \in \xi$.

By the above theorem, the structure ξ is called *initial* with respect to $(X, f_i, (X_i, \xi_i))_{i \in I}$.

We can define subspaces and products. Let (X, ξ) be a FN-space and A be a subset of X. The pair (A, ξ_A) is said to be a *subspace* of (X, ξ) if ξ_A is initial with respect to $(A, i, (X, \xi))$, where i is the inclusion map. For a family $((X_i, \xi_i))_{i \in I}$ of FN-spaces, let $X = \Pi_{i \in I} X_i$ be given. The pair (X, ξ) is called a *product fuzzy nearness space* if ξ is initial with respect to $(X, \pi_i, (X_i, \xi_i))_{i \in I}$, where each $i \in I$, $\pi_i : X \to X_i$ is the projection map.

In the above theorem, if $X_i = X$ and f_i is the identity map for all $i \in I$, then $\xi = \bigcap\{\xi_i : i \in I\}$.

THEOREM 4.2. *(Existence of Final Structures). Suppose Y be a set, $((Y_i, \eta_i))_{i\in I}$ be a family in FN-space, and $g_i : Y_i \to Y$ be a surjective map for all $i \in I$. Then:*

(a) $\eta = \bigcap\{g_i(\eta_i) \mid i \in I\}$ is the finest fuzzy nearness structure on Y with respect to which each g_i is a FN-map.

(b) A map $h : (Y, \eta) \to (Z, \zeta)$ is a FN-map if and only if

$$h \circ g_i : (Y_i, \eta_i) \to (Z, \zeta), \quad i \in I$$

is a FN-map.

Proof. (a) (FN1),(FN3) and (FN4) are easy.

(FN2) If $\mathcal{A}$ is q.c., then there exists $y \in Y$ such that $A_1(y)+A_2(y) > 1$ for any $A_1, A_2 \in \mathcal{A}$. Since each g_i is onto, i.e., $g_i(y_i) = y$, we have $g_i^{-1}(\mathcal{A}) \in \eta_i$.

(FN5) We will show that $g_i^{-1}(cl_\eta B) \subset cl_{\eta_i} g_i^{-1}(B)$. If $y_t \; q \; cl_\eta B$, then $\{y_t, B\} \in \eta$. We have $\{g_i^{-1}(y_t), g_i^{-1}(B)\} \in \eta_i$. Hence $cl_{\eta_i} g_i^{-1}(\mathcal{B}) \prec g_i^{-1}(cl_\eta(\mathcal{B}))$.

Since $g_i^{-1}(g_i(\mathcal{B})) \prec \mathcal{B}$ for every $\mathcal{B} \in \eta_i$, $g_i : (Y_i, \eta_i) \to (Y, \eta)$ is a FN-map for each $i \in I$.

(b) It is easy.

By the above theorem, the structure η is called *final* with respect to $((Y_i, \eta_i), g_i, Y)_{i\in I}$.

Using this structure, we can define quotient spaces. Let (X, ξ) be a FN-space and $g : X \to Y$ is surjective. The pair (Y, η) is said to be a *quotient FN-space* of (X, ξ) if η is final with respect to $((X, \xi), g, Y)$, where g is called a *quotient FN-map*.

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Department of Mathematics
Kangnung National University
Kangnung, 210-702, Korea