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SOME LIMIT THEOREM TO JUMP-DIFFUSIONS IN STOCHASTIC DIFFERENCE EQUATIONS

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0. Introduction.

In order to approach a non-Markovian processes to Markov processes, the matter of what conditions are necessary is very interested and worked by many authors. In this paper, we will study if the sequence of non-Markovian stochastic processes defined by the solution of stochastic difference equations satisfy some conditions, then it converges to jump type process with Markov property. Especially, we will show how the continuous part of limiting process and jump part are derived. Also, by many scholars, the limit theorems are treated in the standpoint of view which is the convergence of characteristic of semimartingales and some other processes [2], [3]. But we will use some mixing property which does not yields the convergence of characteristics.

In a broad sense, this paper is nothing but one of the result obtained by many scholars in the part, because the subject and the methods which we treat herein had already appeared and used. But, through this paper, we try to simplify the conditions and, through detailed computation, to show the course that the result comes out. Therefore, we restrict to the sequences of square integrable random variables. Also, we will omit some parts of proof which is got by [1] easily.

The detail content and results are the followings : for the double indexed square integrable random variables $\{X_k^n \ ; \ n, \ k \in \mathbb{N}\}$ of $\mathbb{R}^e$-valued, the $d \times e$-matrix valued functions F^n defined on $\mathbb{R}^d$, and $\mathbb{R}^d$-valued functions G^n, we consider the stochastic difference equation :

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$$(0.1)\qquad \begin{cases} \varphi_k^n - \varphi_{k-1}^n = F^n(\varphi_{k-1}^n)(X_k^n - a^n) + (\frac{1}{j_n})G^n(\varphi_{k-1}^n), \\ \varphi_0^n = x_0 \in \mathbb{R}^d, \end{cases}$$

where $a^n = E[X_1^n I_{(0,1]}(|X_1^n|)]$.

We define an interpolating process $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ by, for some sequence $\{j_n\}$ diviging to infinity,

$$(0.2)\qquad \varphi^n(t) = \varphi_{[j_n t]}^n \quad \text{for } t \in [0,\infty),$$

where $[t]$ is the integral part of t. Then each $\{\varphi^n(t)\}$ can be regared as a sequence of random variables with values in the Skorohod space $\mathbb{D}_d(= \mathbb{D}([0,\infty),\ \mathbb{R}^d))$ of càdlàg functions. As usual, we equip the space $\mathbb{D}_d$ with, so called, the Skorohod topology [5]. We want to study the weak convergence of this sequence $\{\varphi^n\}_{n\in\mathbb{N}}$. For this aim, our standpoint of discussion is the weak convergence of the driving noise processes

$$(0.3)\qquad X^n(t) = \sum_{k=1}^{[j_n t]}(X_k^n - a^n)$$

to a Lévy process implies that of the system processes $\{\varphi_n\}$ to a Markov process with jumps. For the purpose, we will use the mixing condition and apply so-called martingale method.

In section I, we will give the preliminaries a little. In section II, we will give the statement of results detaily. The main theorem is Theorem I which states that the sequence $\{\varphi^n(t)\}$ of (0.2) converges weakly in $\mathbb{D}_d$ to the solution of a stochastic differential equation of jump type. But to prove this result, it is more convenient to show the weak convergence of the joint processes $\tilde{\varphi}^n(t) = (\varphi^n(t),\ X^n(t))$ instead of $\varphi^n(t)$. Theorem II which states on $\{\tilde{\varphi}^n(t)\}_{n\in\mathbb{N}}$ is a rewritting of Theorem I. Therefore, in section III, we will give a proof only for Theorem II. Since it is long, we will divide it into several steps. First, we will show the tightness of the sequence of stochastic processes and then characterize any limiting process by showing the law of it is the unique solution of a martingale problem. Second, we will treat the removal of localization and trancation.

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I. Preliminaries.

In this section, we will give some definitions and notions to formulate our results. We will also give the definition of strongly uniform mixing arry.

For each $n \in \mathbb{N}$, let $\{X_k^n \ ; \ n, \ k \in \mathbb{N}\}$ be an array of $\mathbb{R}^e$-valued random variables defined on $(\Omega, \ \mathcal{F}, \ p)$. We will assume that this array $\{X_k^n \ ; \ n, \ k \in \mathbb{N}\}$ is stationary ; for each $n \in \mathbb{N}$, the joint law of $(X_1^n, \ X_2^n, \ \cdots \ , \ X_k^n)$ under P is equal to that of $(X_{1+l}^n, \ \cdots \ , \ X_{k+l}^n)$ for all $k, \ l \in \mathbb{N}$. We will say that the array $\{X_k^n\}$ of random variable satisfies the strongly uniform mixing condition with the rate function ψ if

$$\psi(k) = \sup_{n\in\mathbb{N}} \sup_{l\in\mathbb{N}} \sup \left\{ \left| \frac{P(A\cap B)}{P(A)P(B)} - 1 \right| ; \ A \in \mathcal{F}_{1,l}^n, B \in \mathcal{F}_{l+k,\infty}^n, \right.$$
$$\left. \text{and } P(A)P(B) > 0 \right\}$$

converges to 0 as $k \to \infty$, where we set $F_{k,l}^n = \sigma[X_\zeta^n \ ; \ k \le \zeta \le l]$ for $1 \le k \le l < \infty$.

We denote by $\mathbb{D}_d = \mathbb{D}([0,\infty), \ \mathbb{R}^d)$ the space of all right continuous functions with lefthand limit from $[0,\infty)$ to d-dimensional Euclidean space $\mathbb{R}^d$, We equip $\mathbb{D}_d$ with the so-called Skorohod topology for which it is a Polish space.

For a sequence $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ of $\mathbb{D}_d$-valued random variables, we say that $\varphi^n(t)$ converges in law to a $\mathbb{D}_d$-valued random variable $\varphi(t)$ if the law of $\varphi^n(t)$ converge weakly to that of $\varphi(t)$ and we denote $\varphi^n(t) \xrightarrow{\mathcal{L}} \varphi(t)$ in $\mathbb{D}_d$ as $n \to \infty$.

On the other hand, it is well-known that for a σ-finite measure ν on $(\mathbb{R}^e \backslash \{0\}, \beta(\mathbb{R}^e \backslash \{0\}))$ satisfying

$$\int_{\mathbb{R}^e\setminus\{0\}} \min\{|Z|^2,\ 1\}\nu(dz) < \infty\ ,$$

there exists a stationary Poisson point process $\{p(t)\}$ on $\mathbb{R}^e\setminus\{0\}$ with the intensity measure ν. We denote its counting measure by $N_p(dudz)$ and its compensated counting measure by $\tilde{N}_p(dudz)$ $(= N_p(dudz) - du\nu(dz))$.

We denote by $C^2(\mathbb{R}^d,\ \mathbb{R}^e)$ the space of all function from $\mathbb{R}^d$ into possessing continuous derivatives of order up to and including 2. In the case of $e = 1$, we denote it by $C^2(\mathbb{R}^d)$. We also denote by $C_b^2(\mathbb{R}^d,\ \mathbb{R}^e)$ the space of all functions of class $C^2(\mathbb{R}^d,\ \mathbb{R}^e)$ possessing bounded derivatives. $B(\mathbb{R}^d,\ \mathbb{R}^e)$ denotes the space of all bounded measurable functions from $\mathbb{R}^d$ into $\mathbb{R}^e$. We denote $\mathbb{R}^d \otimes \mathbb{R}^e$ the set of all real $d \times e$ matrices, which is indentified with $\mathbb{R}^{d\times e}$. We also denote by $C_0(\mathbb{R}^d)$ the space of all bounded functions defined on $\mathbb{R}^d$ which are 0 arround $0 \in \mathbb{R}^e$ and have a limit at the infinity.

For an array $\{X_k^n\}$ on $\mathbb{R}^e$, we set

$$\begin{aligned} X_{k,\delta}^n &= X_k^n\ I_{(0,\delta)}(|X_k^n|) \\ X_k^{n(\delta)} &= X_k^n\ I_{(\delta,\infty)}(|X_k^n|)\ . \end{aligned}$$

for some $\delta > 0$. We denote by $X_k^{n,p}$ the p-th component of X_k^n for $p = 1,\ 2,\ 3,\ \cdots,e$.

II. Statement of Results.

Let $\{X_k^n\ ;\ n,\ k \in \mathbb{N}\}$ be an array of $(\mathbb{R}^e\setminus\{0\})$-valued, stationary, and square integrable random variables such that the squence integral is converge to 0 when $n \to \infty$. Then we can choose a sequence $\{j_n\}$ diverging to infinity such that $j_n\ E[|X_{1,\delta_0}^n|^2] < \infty$ for some $\delta_0 > 0$. We will write several conditions to get the main results.

$(A.I)$ $j_n P(X_1^n \in dz)$ converge to Lévy measure ν defined on $(\mathbb{R}^e\setminus\{0\}\ ,\ \mathcal{B}(\mathbb{R}^e\setminus\{0\}))$, such that for all $f \in C_0(\mathbb{R}^e)$,

$$\text{(II.1)} \qquad j_n \int_{\mathbb{R}^e} f(z)P(X_1^n \in dz) \longrightarrow \int_{\mathbb{R}^e} f(z)\nu(dz)\ . \quad (n \to \infty)\ .$$

$(A.II)$ There exist real numbers $V_0^{p,q}$ and $V_1^{p,q}$ for all $p,\ q\ = 1,\ 2,\ \cdots\ , e$ such that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\left|\, j_n\ E[\eta_{1,\delta}^{n,p}\eta_{1,\delta}^{n,q}] - V_0^{p,q}\right| = 0 \tag{II.2}$$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\left|\, j_n\ \sum_{k=2}^{j_n} E[\eta_{1,\delta}^{n,p}\eta_{k,\delta}^{n,q}] - V_1^{p,q}\right| = 0 \tag{II.3}$$

where $\eta_{k,\delta}^n = X_{k,\delta}^n - E[X_{k,\delta}^n], X_{k,\delta}^n = X_k^n I_{\{|X_k^n|\le\delta\}}$, and $\eta_k^{n,p}$ is the p-th component of η_k^n.

$(A.III)$ The mixing rate function of the strong uniform mixing array $\{X_k^n\}$ satisfies

$$\sum_{k=1}^{\infty}\sqrt{\psi(k)}\ < \infty. \tag{II.4}$$

Consider the stochastic difference equations

(II.5)
$$\begin{cases} \varphi_k^n - \varphi_{k-1}^n = F^n(\varphi_{k-1}^n)(X_k^n - a^n) + \left(\dfrac{1}{j_n}\right) G(\varphi_{k-1}^n), \ k = 1, 2, \cdots \\ \varphi_0^n = x_0, \end{cases}$$

where we set $a^n = E[X_1^n I_{(0,1]}(|X_1^n|)]$ and I_A denote the inticator function of a set A.

For the sequence $\{j_n\}$ which we took, define a sequence of stochastic processes $\{\varphi^n\}_n$ by

$$\varphi^n(t) = \varphi_{[j_n t]}^n \quad \text{for } t\in[0,\infty) \tag{II.6}$$

such that $\varphi_{[j_n t]}^n$ satisfy $(II-5)$. Then each $\varphi^n(t)$ can be regared as a random variable with values in the Skorohod space $\mathbb{D}_d([0,\infty),\ \mathbb{R}^d)$.

Suppose the coefficients $F^n(x)$ and $G^n(x)$ in $(II-5)$ satisfy

$(C^n.I)$. For a sequence $\{F^n\}_n \in C^2(\mathbb{R}^d,\ \mathbb{R}^d \otimes \mathbb{R}^e)$, there exists a function F satisfying

$$F \in C^2(\mathbb{R}^d,\ \mathbb{R}^d \otimes \mathbb{R}^e) \cap B(\mathbb{R}^d,\ \mathbb{R}^d \otimes \mathbb{R}^e).$$

and

$$\lim_{n\to\infty} \sum_{i=0}^{2} \left(\sup_{|x|\le N} |D^i F^n(x) - F^i(x)| \right) = 0 \tag{II.7}$$

for each $N > 0$.

$(C^n.II)$ For a sequence of continuous functions $\{G^n\}_n$ from $\mathbb{R}^d$ into $\mathbb{R}^d$ itself, there exists a function G satisfying that ;

G is a locally Lipschitz continuous function from $\mathbb{R}^d$ into itself such that

$$\sup_{|x|\le N} |G^n(x) - G(x)| \to 0 \text{ as } n \to \infty \text{ for each } N > 0. \tag{II.8}$$

Then we have the following result.

Theorem I. Assume $(A.I)$, $(A.II)$, $(A.III)$, $(C^n.I)$ and $(C^n.II)$ for $\{X_k^n\}$, F^n and G^n of $(II.5)$. Then the processes $\{\varphi^n(t)\}_n$ of $(II.6)$ converge in law to the unique solution $\varphi(t)$ of the following stochastic differential equation ;

$$\begin{aligned} \varphi(t) = x_0 &+ \int_0^t F(\varphi(u)B^V(du) + \int_0^t (G+b)(\varphi(u))du \\ &+ \int_0^{t+} \int_{|z|\le\gamma} F(\varphi(u-)z\tilde{N}_p(dudz) \\ &+ \int_0^{t+} \int_{|z|>\gamma} F(\varphi(u-)zN_p(dudz). \end{aligned} \tag{II.9}$$

where B^V is Brownian motion with mean 0 and covariant matrix

$$(V^{p,q} = V_0^{p,q} + V_1^{p,q} + V_1^{q,p})_{p,q=1,2,\cdots,e},$$

$p(t)$ is e-dimensional stationary Poisson point process with the intensity measure ν, and

$$b^j(x) = \sum_{i=1}^{d} \sum_{p,q=1}^{e} F_p^i(x) V_1^{p,q} \frac{\partial F_q^i(x)}{\partial x^i}.$$

If we define the driving noise processes as

$$X^n(t) = \sum_{k=1}^{[j_n t]} (X_k^n - a^n), \tag{II.10}$$

then note that by the representation of $(II.5)$ and $(II.9)$, we can see the prelimiting process $\varphi^n(t)$ and the limiting process $\varphi(t)$ are functionals of $(\varphi^n(t), X^n(t))$ and $(\varphi(t), X(t))$ respectively. Here

(II.11)

$$X(t) = B^V(t) + \int_0^{t+} \int_{|z|\le\gamma} z\tilde{N}_p(dudz) + \int_0^{t+} \int_{|z|>\gamma} zN_p(dudz).$$

Therefore, for the proof of Theorem I, we need to show the weak convergence of $\{X^n(t)\}_n$ as well as that of $\{\varphi^n(t)\}_n$. To this end, it is sufficient to show the weak convergence of the pair $\tilde{\varphi}^n(t) = (\varphi^n(t), X^n(t))$ in the product space $\mathbb{D}_d \times \mathbb{D}_e$. But we will give a stronger assertion ; the weak convergence of $\{\tilde{\varphi}^n(t)\}_n$ in $\mathbb{D}_{d+e}$.

Theorem II. Let $\varphi^n(t)$ and $X^n(t)$ be the processes defined in $(II.6)$ and $(II.10)$, respectively. Set

$$\tilde{\varphi}^n(t) = \begin{pmatrix} \varphi^n(t) \\ X^n(t) \end{pmatrix}.$$

Then, under the assumptions $(A.I)$, $(A.II)$, $(A.III)$, $(C^n.I)$ and $(C^n.II)$, it holds

$$\tilde{\varphi}^n(t)\xrightarrow{\mathcal{L}}\tilde{\varphi}(t) \quad \text{in } \mathbb{D}_{d+e} \quad \text{as } n \to \infty,$$

where $\tilde{\varphi}(t)$ is the unique solution of the following stochastic differential equation ;

$$\begin{aligned} \tilde{\varphi}(t) = \tilde{x}_0 &+ \int_0^t \tilde{F}(\tilde{\varphi}(u)B^V(du) + \int_0^t (\tilde{G}+\tilde{b})(\tilde{\varphi}(u))du \\ &+ \int_0^{t+} \int_{|z|\leq\gamma} \tilde{F}(\tilde{\varphi}(u-)z\tilde{N}_p(dudz) \\ &+ \int_0^{t+} \int_{|z|>\gamma} \tilde{F}(\tilde{\varphi}(u-)zN_p(dudz). \end{aligned} \tag{II.12}$$

Here, we put $\tilde{x}_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \in \mathbb{R}^{d+e}$ for $x_0 \in \mathbb{R}^d$ and $0 \in \mathbb{R}^e$,

$$\tilde{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F(x) \\ I_e \end{pmatrix}$$; where I_e is the e × e-unit matrix,

$$\tilde{G}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} G(x) \\ 0 \end{pmatrix}$$

$$\tilde{b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b(x) \\ 0 \end{pmatrix}$$ for $x \in \mathbb{R}^b$ and $y \in \mathbb{R}^e$.

Remark 1. In [2], we can meet the weak convergence of the sequence $\{\varphi^n\}$ drived by the solution of stochastic difference equations by the method of convergence of characteristics. But this is not same result and we don't know the result unifying both cases of driving processes in [2] and some our mixing sequence method yet.

Remark 2. In [3], we can meet some stochastic differential equation of the form ;

$$\frac{d}{dt}\varphi_t = f_t^n(\varphi_t) + \bar{g}_t^n(\varphi_t).$$

where $f_t^n = f_t^n(x), t \geq 0$ is a stochastic process and $\bar{g}_t^n = \bar{g}_t^n(x), t \geq 0$ is a deterministic functions. In here, if we put

$$f_u^n(\varphi_{u-}^n) = F^n(\varphi_{u-}^n)X_u^n$$
$$\bar{g}_u^n(\varphi_{u-}^n) = \left(\frac{1}{j_n}\right) G^n(\varphi_{u-}^n) - F^n(\varphi_{u-}^n)a^n,$$

where $a^n = E[X_1^n I_{\{|X_1^n| \leq \gamma\}}(|X_1^n|)]$, then we can drive a stochastic difference equation of the form

$$\varphi_k - \varphi_{k-1} = f_{k-1}^n(\varphi_{k-1})X_{k-1}^n + g_{k-1}^n(\varphi_{k-1}), \quad k = 1, 2, \cdots .$$

Also, an unified method which is applicable both to stochastic differential equation and to stochastic difference equation is introduced as a stochastic difference-differential of the form ;

$$\varphi_t^n = x_0 + \int_0^t [f_u^n(\varphi_{u-}^n) + \bar{g}_u^n(\varphi_{u-}^n)]dA_u^n,$$

where $\{f_u^n\}$ is a sequence of vector field valued stochastic processes, $\{\bar{g}_u^n\}$ is a sequence of deterministic vector field valued functions of u, and $\{A_u^n\}$ is a sequence of deterministic nondecreasing càdlàg functions of u. The limit theorems for this stochastic difference-differential equations are given by another method in [3].

III. Proof the Results.

In this section, we will prove Theorem II only, because Theorem I is belonged in Theorem II. We will divide the proof by serveral steps because it is very long. The 1st step is to establish the weak convergence by the martingale method for the localized and truncated process of $\{\varphi^n(t)\}_{n\in\mathbb{N}}$, which are uniformly bounded. To complete the 1st step, we show the tightness and then characterize any limiting processes. The 2nd step is to remove the restriction of localization and truncation which complete the proof of Theorem II.

$III-1$. Localized and Truncated Processes.

For each $L > 0$, let $\gamma_L(x)$ be a smooth function on $\mathbb{R}^d$ such that

$$\gamma_L(x) = \begin{cases} 1, & \text{if } |x| \le L \\ 0, & \text{if } |x| \ge L+1, \end{cases}$$

and let $h_\gamma(x)$ be also another smooth function on $\mathbb{R}^e$ such that

$$h_L(x) = \begin{cases} 1, & \text{if } |x| \le L \\ 0, & \text{if } |x| \ge L+1, \end{cases}$$

Set

$$F_L^n(x) = \gamma_L(x) F^n(x)$$
$$G_L^n(x) = \gamma_L(x) G^n(x)$$

and for each $M \in C(\nu) := \{r > 0; \nu(\{z; |z| = r\}) = 0\}$ and $L > 0$, we define

$$\text{(III.1)} \qquad \begin{cases} \varphi_{k,M,L}^n - \varphi_{k-1,M,L}^n = F_L^n(\varphi_{k-1,M,L}^n)(X_{k,M}^n - a^n) \\ \qquad\qquad\qquad\qquad + \left(\dfrac{1}{j_n}\right) G_L^n(\varphi_{k-1,M,L}^n) \\ \varphi_{0,M,L}^n = x_0 \end{cases}$$

Assume that $L \ge 2|x_0|$, and similarly for some function $h_L^n(x)$, we define

$$\text{(III.2)} \qquad \begin{cases} \zeta_{k,M,L}^n - \zeta_{k-1,M,L}^n = h_L^n(\zeta_{k-1,M,L}^n)(X_{k,M}^n - a^n) \\ \zeta_{0,M,L}^n = 0 \end{cases}$$

Set

$$\text{(III.3)} \qquad \tilde{\varphi}_{M,L}^n(t) = \begin{pmatrix} \varphi_{M,L}^n(t) \\ \zeta_{M,L}^n(t) \end{pmatrix} = \begin{pmatrix} \varphi_{[j_n t],M,L}^n \\ \zeta_{[j_n t],M,L}^n \end{pmatrix}.$$

We call this processes the localized and truncated processes of $\tilde{\varphi}^n(t)$. Note that $\tilde{\varphi}_{M,L}^n(t)$ are uniformly bounded and has uniform bounded jumps, that is, there exists a constant $K_{M,L} > 0$ such that

$$\sup_{n,t,w} |\tilde{\varphi}^n_{M,L}(t,w)| < K_{M,L},$$
$$\sup_{n,t,w} |\Delta\tilde{\varphi}^n_{M,L}(t,w)| < K_{M,L}$$

where $\Delta\varphi(t) = \varphi(t) - \varphi(t-)$.

$III-1.(a)$ Tightness of $\{\tilde{\varphi}^n_{M,L}(t)\}_n$.

First, we will give a Lemma on the strongly uniform mixing property, which will be used frequently in the proof.

Lemma $III.1$. Set $m < l < k$.

(1) For an $\mathcal{F}^n_{1,l}$-measurable integrable function X and for an $\mathcal{F}^n_{k,l}$-measurable integrable function Y, it holds

$$|E(XY) - E(X)E(Y)| \leq \psi(k-l)E[|X|]E[|Y|].$$

(2) Let X be an $\mathcal{F}^n_{1,m}$-measurable integrable function, Y be an $\mathcal{F}^n_{l,l}$-measurable integrable function and Z be an $\mathcal{F}^n_{k,\infty}$-measurable integrable function with $E[Z]=0$. Then it holds

$$|E[X(YZ - E(YZ))]| \leq \sqrt{2}(\psi^* + 1)\sqrt{\psi(k-l)}\sqrt{\psi(l-m)}$$
$$E[|X|]E[|Y|]E[|Z|],$$

where $\psi^* = \psi(1) + 1$.

We will omit this proof, because we can meet it in [1].

To get the tightness of the sequence $\{\tilde{\varphi}^n_{M,L}(t)\}_n$, we will show that the sequence satisfies the Kolmogorov-Chentsov's criterion. This following proposition implies the tightness of $\{\tilde{\varphi}^n_{M,L}(t)\}_n$ since $\tilde{\varphi}^n_{M,L}(0) = \tilde{x}_0$.

Proposition $III.1$. For each $T > 0$, there exists a constants $K > 0$ such that

(III.4)
$$E[|\tilde{\varphi}^n_{M,L}(t) - \tilde{\varphi}^n_{M,L}(s)|^2|\tilde{\varphi}^n_{M,L}(s) - \tilde{\varphi}^n_{M,L}(r)|^2] \leq K|t-r|^2$$

for all $0 \leq r \leq s \leq t \leq T$, and

$$\text{(III.5)} \qquad E[|\tilde{\varphi}^n_{M,L}(t) - x_0|^2 \le Kt \qquad \text{for all} \quad t \le T,$$

Proof. Before going to the proof, we give a few remarks to make the notations simple. First, we take $j_n = n$ throughout section III. Next we put

$$\tilde{F}^n_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F^n_L(x) \\ h^n_L(y) \end{pmatrix} \qquad \text{for} \quad x \in \mathbb{R}^d \quad \text{and} \quad y \in \mathbb{R}^e,$$

$$\tilde{G}^n_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} G^n_L(x) \\ q^n_L(y) \end{pmatrix} \qquad \text{for} \quad x \in \mathbb{R}^d \quad \text{and} \quad y \in \mathbb{R}^e.$$

Then $\tilde{\varphi}^n_{M,L}(t)$ of $(III.3)$ is represented as

(III.6)

$$\tilde{\varphi}^n_{M,L}(t) = \tilde{x}_0 + \sum_{k=1}^{[nt]} \tilde{F}^n_L(\tilde{\varphi}^n_{k-1,M,L})(\tilde{X}^n_{k,M} - a^n) + \sum_{k=1}^{[nt]} \left(\frac{1}{j_n}\right) \tilde{G}^n_L(\tilde{\varphi}^n_{k-1,M,L}),$$

where we set

$$\tilde{\varphi}^n_{k,M,L} = \begin{pmatrix} \varphi_{k,M,L} \\ \zeta_{k,M,L} y \end{pmatrix}$$

Since it is similar to the equation $(III,1)$, we may replace $\tilde{F}^n_L$ and $\tilde{\varphi}^n_{M,L}$ of $(III.6)$ by F^n_L and $\varphi^n_{M,L}$ respectively, because we do not need any changes of discussion. In the sequel, we omit subscripts M and L in $\varphi^n_{M,L}$ and $\varphi_{M,L}$, and subscript L in F^n_L. We will also give proofs of $(III.4)$ and $(III.5)$ only in 1 -dimensional case. Further, by the uniform boundedness of $\{\varphi^n_{M,L}(x)\}_n$, we may assume $F^n \in C^2_b(\mathbb{R}^1, \mathbb{R}^1)$. For fixed $M > \gamma$, put

$$\eta^n_k = X^n_{k,M} - E[X^n_{k,M}]$$
$$b^n = E[{X^n_{k,M}}^{(1)}] = E[{X^n_{1,M}}^{(1)}].$$

Then we have

$$
\begin{aligned}
&|\varphi^n(t) - \varphi^n(s)|^2 \\
&= \left| \sum_{k=1}^{[nt]} F^n(\varphi_{k-1}^n)(\eta_k^n + b^n) + \left(\frac{1}{n}\right) \sum_{k=1}^{[nt]} G^n(\varphi_{k-1}^n) \right. \\
&\quad \left. - \sum_{k=1}^{[ns]} F^n(\varphi_{k-1}^n)(\eta_k^n + b^n) - \left(\frac{1}{n}\right) \sum_{k=1}^{[ns]} G^n(\varphi_{k-1}^n) \right|^2 \\
&= \left| \sum_{k=[ns]+1}^{[nt]} F^n(\varphi_{k-1}^n)(\eta_k^n + b^n) + \left(\frac{1}{n}\right) \sum_{k=[ns]+1}^{[nt]} G^n(\varphi_{k-1}^n) \right|^2 \\
&\le 2 \left| \sum_{k=[ns]+1}^{[nt]} F^n(\varphi_{k-1}^n)(\eta_k^n + b^n) \right|^2 + 2\left(\frac{1}{n}\right)^2 \left| \sum_{k=[ns]+1}^{[nt]} G^n(\varphi_{k-1}^n) \right|^2 \\
&\le 4 \sum_{k=[ns]+1}^{[nt]} (F^n(\varphi_{k-1}^n))^2 (\eta_k^n)^2 + 8 \sum_{l<k} F^n(\varphi_{l-1}^n)\eta_l^n F^n(\varphi_{k-1}^n)\eta_k^n \\
&\quad + 4 \left\{ \sum_{k=[ns]+1}^{[nt]} F^n(\varphi_{k-1}^n) b^n \right\}^2 + 2\left(\frac{1}{n}\right)^2 \left\{ \sum_{k=[ns]+1}^{[nt]} G^n(\varphi_{k-1}^n) \right\}^2 \\
&=: 4I_1^n + 8I_2^n + 4I_3^n + 2\left(\frac{1}{j_n}\right)^2 I_4^n.
\end{aligned}
$$

In the above, $\sum_{l<k}$ denotes the summation ober (l, k) such that $[ns] + 1 \le l \le k \le [nt]$. We will use this abbreviation later too. We will show the following lemma.

Lemma III.2. It holds that

$$
\left| E[I_i^n |\varphi^n(s) - \varphi^n(r)|^2] \right| \le K \frac{[nt] - [ns]}{n} E[|\varphi^n(s) - \varphi^n(r)|^2] \qquad \text{for} \quad i = 1, 2, 3, 4
$$

where K is a constant which does not depend on n, r, s and t.

Proof. We first consider I_1^n. Note that from assumptions and $(II.1)$, it holds

$$\sup_{n,k} nE[|\eta_k^n|^2] =: K_1 < \infty. \tag{III.7}$$

Here, we have

$$\begin{aligned}
&E[I_1^n|\varphi^n(s)-\varphi^n(r)|^2] \\
&\quad = E\left[\sum_{k=[ns]+1}^{[nt]} \left(F^n(\varphi_{k-1}^n)\right)^2 (\eta_k^n)^2|\varphi^n(s)-\varphi^n(r)|^2\right] \\
&\quad = \sum_{k=[ns]+1}^{[nt]} E\left[\left(F^n(\varphi_{k-1}^n)\right)^2 (\eta_k^n)^2|\varphi^n(s)-\varphi^n(r)|^2\right] \\
&\quad \le [\psi(1)+1] \sum_{k=[ns]+1}^{[nt]} E\left[\left(F^n(\varphi_{k-1}^n)\right)^2 |\varphi^n(s)-\varphi^n(r)|^2\right] E[|\eta_k^n|^2] \\
&\quad \le [\psi(1)+1]\|F\|^2([nt]-[ns])E[|\varphi^n(s)-\varphi^n(r)|^2]E[|\eta_k^n|^2] \\
&\quad = [\psi(1)+1]\|F\|^2\frac{([nt]-[ns])}{n}\cdot nE[|\eta_k^n|^2]E[|\varphi^n(s)-\varphi^n(r)|^2] \\
&\quad \le [\psi(1)+1]\|F\|^2K_1\frac{([nt]-[ns])}{n}E[|\varphi^n(s)-\varphi^n(r)|^2].
\end{aligned}$$

Here, we denote by $\|F\| = \sup_{n,x}|F^n(x)|$. For I_2^n, we devide it into the sum ;

$$\begin{aligned}
I_2^n &= \sum_{l<k} F^n(\varphi_{l-1}^n)\eta_l^n F^n(\varphi_{k-1}^n)\eta_k^n \\
&= \sum_{l<k}(F^n(\varphi_{l-1}^n))^2\eta_l^n\eta_k^n + \sum_{l<k} F^n(\varphi_{l-1}^n)\eta_l^n\{F^n(\varphi_{k-1}^n)-F^n(\varphi_{l-1}^n)\}\eta_k^n \\
&=: I_{21}^n + I_{22}^n.
\end{aligned}$$

As for the first term, from Lemma III.1.(1), we obtain

(III.8)

$$
\begin{aligned}
&|E[I_{21}^n|\varphi^n(s)-\varphi^n(r)|^2]| \\
&= \left| E\left[\sum_{l<k} (F^n(\varphi_{l-1}^n))^2 \eta_l^n \eta_k^n |\varphi^n(s)-\varphi^n(r)|^2 \right] \right| \\
&\leq \sum_{l<k} [\psi(1)+1] E[|(F^n(\varphi_{l-1}^n))^2 \eta_l^n |\varphi^n(s)-\varphi^n(r)|^2|] E[|\eta_k^n|] \\
&= \sum_{l<k} [\psi(1)+1] E[|F^n(\varphi_{l-1}^n)|^2 |\eta_l^n| |\varphi^n(s)-\varphi^n(r)|^2] E[|\eta_k^n|] \\
&\leq \|F\|^2 [\psi(1)+1] \\
&\qquad \left(\sum_{k=1}^{\infty} \psi(k) \right) \frac{([nt]-[ns])}{n} \cdot nE[|\eta_k^n|^2] E[|\varphi^n(s)-\varphi^n(r)|^2] \\
&\leq \|F\|^2 [\psi(1)+1] \\
&\qquad \left(\sum_{k=1}^{\infty} \psi(k) \right) K_1 \frac{([nt]-[ns])}{n} \cdot E[|\varphi^n(s)-\varphi^n(r)|^2].
\end{aligned}
$$

On the second term, from the mean value theorem and $(III.6)$, we have for each k,

$$
\begin{aligned}
&\sum_{l=[ns]+1}^{k-1} F^n(\varphi_{l-1}^n)\eta_l^n \{F^n(\varphi_{k-1}^n) - F^n(\varphi_{l-1}^n)\}\eta_k^n \\
&= \sum_{l=[ns]+1}^{k-1} F^n(\varphi_{l-1}^n)\eta_l^n \left(\sum_{j=l}^{k-1} \{F^n(\varphi_j^n) - F^n(\varphi_{j-1}^n)\} \right) \eta_k^n \\
&= \sum_{l=[ns]+1}^{k-1} F^n(\varphi_{l-1}^n)\eta_l^n \left(\sum_{j=l}^{k-1} \{\varphi_j^{n*}(\varphi_j^n - \varphi_{j-1}^n)\} \right) \eta_k^n
\end{aligned}
$$

where we put $\displaystyle \varphi_j^{n*} = \int_0^1 F^{n'}(\varphi_{j-1}^n + \theta(\varphi_j^n - \varphi_{j-1}^n))d\theta,$

Therefore it holds by the same method as $(III.8)$,

(III.9)

$$
\begin{aligned}
&|E[I^n_{22}|\varphi^n(s)-\varphi^n(r)|^2]| \\
&=\left|E\left[\sum_{l<k}F^n(\varphi^n_{l-1})\eta^n_l\{F^n(\varphi^n_{k-1})-F^n(\varphi^n_{l-1})\}\eta^n_k|\varphi^n(s)-\varphi^n(r)|^2\right]\right| \\
&\leq\sum_{l<k}\psi(k-l)E[|F^n(\varphi^n_{l-1})\eta^n_l\{F^n(\varphi^n_{k-1})-F^n(\varphi^n_{l-1})\}|\varphi^n(s)-\varphi^n(r)|^2|] \\
&\qquad\qquad E[|\eta^n_k|] \\
&\leq\sum_{l<k}\psi(k-l)E[|F^n(\varphi^n_{l-1})||\eta^n_l||F^n(\varphi^n_{k-1})-F^n(\varphi^n_{l-1})||\varphi^n(s)-\varphi^n(r)|^2] \\
&\qquad\qquad E[|\eta^n_k|] \\
&\leq\|F\|[\psi(1)+1]\left(\sum_{k=1}^{\infty}\psi(k)\right)\frac{([nt]-[ns])}{n}\cdot nE[|\eta^n_1|^2] \\
&\qquad\left(\sum_{j=[ns]+1}^{[nt]}|\varphi^{n*}_j(\varphi^n_j-\varphi^n_{j-1})|\right)E[|\varphi^n(s)-\varphi^n(r)|^2] \\
&\leq K_1\cdot K_2\|F\|[\psi(1)+1]\left(\sum_{k=1}^{\infty}\psi(k)\right)\frac{([nt]-[ns])}{n}\cdot E[|\varphi^n(s)-\varphi^n(r)|^2],
\end{aligned}
$$

because the summand

$$\sum_{j=[ns]+1}^{[nt]}|\varphi^{n*}_j(\varphi^n_j-\varphi^n_{j-1})|$$

is $\mathcal{F}^n_{1,j}$-measurable and is dominated by K_2 which does not depend on n and j because of the uniform boundedness of $\{\varphi^n_j\}$.

From $(III.8)$ and $(III.9)$, we get that

$$E[I^n_2|\varphi^n(s)-\varphi^n(r)|^2]\leq K\frac{([nt]-[ns])}{n}\cdot E[|\varphi^n(s)-\varphi^n(r)|^2].$$

We consider I^n_3. Since

$$\left| \sum_{k=[ns]+1}^{[nt]} F^n(\varphi_{k-1}^n) b^n \right| \leq \|F\| \frac{([nt]-[ns])}{n} \cdot K_3,$$

it holds ;

$$\begin{aligned} &|E[I_3^n|\varphi^n(s)-\varphi^n(r)|^2]| \\ &= \left| E \left[\left| \sum_{k=[ns]+1}^{[nt]} F^n(\varphi_{k-1}^n) b^n \right|^2 |\varphi^n(s)-\varphi^n(r)|^2 \right] \right| \\ &\leq T(K_3)^2 \|F\|^2 \frac{([nt]-[ns])}{n} \cdot E[|\varphi^n(s)-\varphi^n(r)|^2]. \end{aligned} \tag{III.10}$$

Finally, we consider I_4^n. By the same method as I_3^n, we get the result.

(III.11)

$$|E[I_4^n|\varphi^n(s)-\varphi^n(r)|^2]| \leq K_5 \frac{([nt]-[ns])}{n} \cdot E[|\varphi^n(s)-\varphi^n(r)|^2].$$

We have now completed the proof of Lemma III.2.

We continue the proof of Proposition III.1. By Lemma III.2, we get the estimate

$$\begin{aligned} &E[|\varphi^n(t)-\varphi^n(s)|^2|\varphi^n(s)-\varphi^n(r)|^2] \\ &\qquad \leq K_6 \frac{([nt]-[ns])}{n} \cdot E[|\varphi^n(s)-\varphi^n(r)|^2]. \end{aligned}$$

for some constant $K_6 > 0$.

Clearly, we also have

$$E[|\varphi^n(s)-\varphi^n(r)|^2] \leq K_6 \frac{([ns]-[nr])}{n}. \tag{III.12}$$

These yield

$$E[|\varphi^n(t)-\varphi^n(s)|^2|\varphi^n(s)-\varphi^n(r)|^2] \le (K_6)^2\left(\frac{([nt]-[ns])}{n}\right)^2,$$

which implies $(III.4)$. $(III.5)$ is obtained from $(III.12)$ taking $s=t$ and $r=0$. We have now completed the proof of Proposition III.1 .

$III-1.(b)$ Characterization of limiting process.

To show the identification of any limit measure of $\{\tilde{P}^n_{M,L}=$ the law of $\tilde{\varphi}^n_{M,L}\}_n$, we establish a proposition.

Proposition III.2. Let $\tilde{P}_{M,L}$ be any limit measure of $\{\tilde{P}^n_{M,L}\}_n$. Define

$$\begin{aligned}\tilde{\mathcal{L}}_{M,L}f(\tilde{x}) &= \frac{1}{2}\sum_{i,j=1}^{d+e}(\tilde{F}_L(\tilde{x})V\tilde{F}_L(\tilde{x})^*)^{i,j}\frac{\partial^2 f}{\partial\tilde{x}^i\partial\tilde{x}^j}(\tilde{x})\\ &+\sum_{j=1}^{d+e}(\tilde{G}^j_L(\tilde{x})+\tilde{b}^j_L(\tilde{x}))\frac{\partial f}{\partial\tilde{x}^j}(\tilde{x})\\ &+\int_{|z|\le M}\Big\{f(\tilde{x}+\tilde{F}_L(\tilde{x})z)-f(\tilde{x})\\ &-\sum_{j=1}^{d+e}(\tilde{F}_L(\tilde{x})z)^j I_{\{|z|\le 1\}}\frac{\partial f}{\partial\tilde{x}^j}(\tilde{x})\Big\}\,\nu(dz),\end{aligned}$$

where

$$\tilde{b}^j_L(\tilde{x})=\sum_{i=1}^{d+e}\sum_{p,q=1}^{e}\tilde{F}^i_{L,p}(\tilde{x})V_1^{p,q}\frac{\partial\tilde{F}^i_{L,q}}{\partial\tilde{x}^i}(\tilde{x}).$$

Let

$$f(\tilde{x})=\exp(i\tilde{u}\,.\,\tilde{x}),$$

where $\tilde{x},\tilde{u}\in\mathbb{R}^{d+e}$, and $\tilde{u}\,.\,\tilde{x}=\sum_{j=1}^{d+e}\tilde{u}^j\,.\,\tilde{x}^j$. Then

(III.13) $$M_f(t) = f(\tilde{\varphi}(t)) - f(\tilde{x}_0) - \int_0^t \tilde{\mathcal{L}}_{M,L} f(\tilde{\varphi}(u))du$$

is a $(\mathbb{D}_{d+e}, \mathcal{D}_t, \tilde{P}_{M,L})$-martingale, where $\mathcal{D}_t$ is the right continuous version of $\sigma[\tilde{\varphi}(u), u \leq t]$.

Proof. We give a proof in 1-dimensional case and omit the subscript M, L and $\sim$. We also assume $F^n, F \in C_b^2(\mathbb{R}^1, \mathbb{R}^1)$. For the limiting measure $P(= \tilde{P}_{M,L})$, set

$$J(\varphi) = \{t > 0 \quad | \quad P(\Delta\varphi(t) \neq 0) > 0\},$$

which is at most countable. To prove this proposition, it suffice to show

(III.14) $$E[\{M_f(t) - M_f(s)\}\Psi(\varphi(u_1), \varphi(u_2), \cdots, \varphi(u_m))] = 0$$

for all $s, t \in J(\varphi)^c, m \in \mathbb{N}, u_i \in J(\varphi)^c (i = 1, 2, \cdots, m), 0 \leq u_1 \leq u_2 \leq \cdots \leq u_m \leq s$, and bounded continuous functions $\Psi; \mathbb{R}^m \to \mathbb{R}$.

In the sequel, we may assume that the law of φ^n converges weakly to $P(= \tilde{P}_{M,L})$. Now, for $s < t$, we have by Taylor's expansion and $(III.6)$, we get

(III.15)

$$
\begin{aligned}
& f(\varphi^n(t)) - f(\varphi^n(s)) \\
&= f(\varphi^n_{[nt]}(t)) - f(\varphi^n_{[ns]}(s)) \\
&= \sum_{k=[ns]+1}^{[nt]} \{f(\varphi^n_k) - f(\varphi^n_{k-1})\} \\
&= \sum_{k=[ns]+1}^{[nt]} f'(\varphi^n_{k-1})(\varphi^n_k - \varphi^n_{k-1}) + \sum_{k=[ns]+1}^{[nt]} (\varphi^{n**}_k)(\varphi^n_k - \varphi^n_{k-1})^2 \\
&= \sum f'(\varphi^n_{k-1}) \left[F^n(\varphi^n_k)\eta^n_k + \left(\frac{1}{n}\right) G^n(\varphi^n_{k-1}) \right] \\
&+ \sum f'(\varphi^n_{k-1}) F^n(\varphi^n_{k-1}) b^n \\
&+ \sum (\varphi^{n**}_k) \left[F^n(\varphi^n_{k-1})(\eta^n_k + b^n) + \left(\frac{1}{n}\right) G^n(\varphi^n_{k-1}) \right]^2 \\
&=: S_1(n) + S_2(n) + S_3(n),
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi^{n**}_k &= \int_0^1 \int_0^1 \alpha f''(\varphi^n_{k-1} + \alpha\beta(\varphi^n_k - \varphi^n_{k-1})) d\alpha \, d\beta \\
&= \int_0^1 \int_0^1 \alpha f'' \left(\varphi^n_{k-1} + \alpha\beta \left[F^n(\varphi^n_{k-1})(\eta^n_k + b^n) \right.\right. \\
&\qquad \left.\left. + \left(\frac{1}{n}\right) G^n(\varphi^n_{k-1}) \right] \right) d\alpha \, d\beta.
\end{aligned}
$$

For some time, we will concentrate on $S_1(n)$. Again by applying Taylor's expansion of $f'F^n$ and by $(III.6)$, we get

(III.16)

$$
\begin{aligned}
S_1(n) &= \sum_{k=[ns]+1}^{[nt]} f'(\varphi_{k-1}^n)\left[F^n(\varphi_{k-1}^n)\eta_k^n + \left(\frac{1}{n}\right) G^n(\varphi_{k-1}^n)\right] \\
&= \sum \Big[f'(\varphi_{k-1}^n)F^n(\varphi_{k-1}^n)\eta_k^n - f'(\varphi^n(s))F^n(\varphi^n(s))\eta_k^n \\
&\qquad + f'(\varphi^n(s))F^n(\varphi^n(s))\eta_k^n + f'(\varphi_{k-1}^n)\left(\frac{1}{n}\right) G^n(\varphi_{k-1}^n)\Big] \\
&= \sum_{k=[ns]+1}^{[nt]} \left[\sum_{l=[ns]+1}^{k-1} (f'(\varphi_l^n)F^n(\varphi_l^n) - f'(\varphi_{l-1}^n)F^n(\varphi_{l-1}^n))\eta_k^n\right] \\
&\qquad + \sum_{k=[ns]+1}^{[nt]} f'(\varphi^n(s))F^n(\varphi^n(s))\eta_k^n \\
&\qquad + \sum_{k=[ns]+1}^{[nt]} f'(\varphi_{k-1}^n)\left(\frac{1}{n}\right) G^n(\varphi_{k-1}^n) \\
&= \sum_{l<k} (f'F^n)'(\varphi_l^n - \varphi_{l-1}^n)\eta_k^n \\
&\qquad + \sum_{l<k} \int_0^1 \int_0^1 \alpha (f'F^n)''[\varphi_{l-1}^n + \alpha\beta(\varphi_l^n - \varphi_{l-1}^n)]d\alpha d\beta \\
&\qquad\qquad (\varphi_l^n - \varphi_{l-1}^n)^2 \eta_k^n \\
&\qquad + \sum_{k=[ns]+1}^{[nt]} f'(\varphi^n(s))F^n(\varphi^n(s))\eta_k^n \\
&\qquad + \sum_{k=[ns]+1}^{[nt]} f'(\varphi_{k-1}^n)\left(\frac{1}{n}\right) G^n(\varphi_{k-1}^n) \\
&= \sum_{l<k} (f'F^n)'F^n(\varphi_{l-1}^n)\eta_l^n\eta_k^n + \sum_{l<k} (f'F^n)'F^n(\varphi_{l-1}^n)b^n\eta_k^n \\
&\qquad + \sum_{l<k} \int_0^1 \int_0^1 \alpha (f'F^n)''[\varphi_{l-1}^n + \alpha\beta(\varphi_l^n - \varphi_{l-1}^n)]d\alpha d\beta \\
&\qquad\qquad (\varphi_l^n - \varphi_{l-1}^n)^2 \eta_k^n \\
&\qquad + \sum_{k=[ns]+1}^{[nt]} (f'F)(\varphi^n(s))\eta_k^n + \sum_{l<k} (f'F)'\left(\frac{1}{n}\right) G^n(\varphi_{l-1}^n)\eta_k^n \\
&\qquad + \sum_{k=[ns]+1}^{[nt]} f'(\varphi_{k-1}^n)\left(\frac{1}{n}\right) G^n(\varphi_{l-1}^n)
\end{aligned}
$$

As for these terms, the following lemma holds.

Lemma III.3. If we put, for Ψ and $u_1, u_2, \cdots, u_m$ in $(III.14)$,

$$\Psi^n(s) = \Psi(\varphi^n(u_1), \varphi^n(u_2), \cdots, \varphi^n(u_m)).$$

Then we get

(1)
$$\limsup_{n\to\infty} \sum_{l<k} |E[(f'(\varphi_l^n)F^n(\varphi_l^n))'F^n(\varphi_{l-1}^n)\{\eta_l^n\eta_k^n - E[\eta_l^n\eta_k^n]\}\Psi^n(s)]| = 0.$$

(2)
$$\limsup_{n\to\infty} \sum_{l<k} |E[(f'(\varphi_l^n)F^n(\varphi_l^n))'b^n\eta_k^n\Psi^n(s)]| = 0.$$

(3)
$$\limsup_{n\to\infty} \sum_{l<k} \left| E\left[\int_0^1 \int_0^1 \alpha(f'F^n)'' \ \ (\varphi_{l-1}^n + \alpha\beta(\varphi_l^n - \varphi_{l-1}^n))d\alpha d\beta \right.\right. \\ \left.\left. (\varphi_l^n - \varphi_{l-1}^n)^2\eta_k^n\Psi^n(s)\right] \right| = 0.$$

(4)
$$\limsup_{n\to\infty} \sum_{k=[ns]+1}^{[nt]} |E[(f'F^n)(\varphi^n(s)\eta_k^n\Psi^n(s)]| = 0.$$

(5)
$$\limsup_{n\to\infty} \sum_{l<k} \left| E\left[(f'F^n)'\left(\frac{1}{n}\right) G^n(\varphi_{l-1}^n)\eta_k^n\Psi^n(s) \right] \right| = 0.$$

(6)
$$\limsup_{n\to\infty} \sum_{k=[ns]+1} \left| E\left[(f'(\varphi_{k-1}^n)\left(\frac{1}{n}\right) G^n(\varphi_{k-1}^n)\Psi^n(s) \right] \right| = 0.$$

Furthermore, the limit of the term in (1) is as following

$$\begin{aligned}(7)\qquad &\lim_{n\to\infty}\sum_{l<k}E[(f'F^n)'F^n(\varphi_{l-1}^n)\Psi^n(s)]E[\eta_l^n\eta_k^n]\\ &=E\left[\int_s^t(f'F^n)'F(\varphi^n(u))du\Psi^n(s)\right]V_1,\end{aligned}$$

where V_1 is the constant defined in $(II.3)$ for 1-dimensional case.

Proof. (1) Note that

$$\begin{aligned}(f'F^n)'F^n(\varphi_{l-1}^n)&=\{(f'F^n)'F^n(\varphi_{l-1}^n)-(f'F^n)'F^n(\varphi^n(s))\}\\ &\quad+(f'F^n)'F^n(\varphi^n(s))\\ &=\sum_{j=[ns]+1}^{l-1}\Phi_j^n(\varphi_j^n-\varphi_{j-1}^n)+(f'F^n)'F^n(\varphi^n(s)),\end{aligned}$$

where

$$\Phi_j^n=\int_0^1[(f'F^n)'F^n]'(\varphi_{j-1}^n+\theta(\varphi_j^n-\varphi_{j-1}^n))d\theta,$$

which is uniformly bounded $\mathcal{F}_{1,j}^n$-measurable.

Then we obtain from Lemma III-1.(2) ;

$$\sum_{l<k} |E[(f'F^n)'F^n(\varphi_{l-1}^n)\{\eta_l^n\eta_k^n - E[\eta_l^n\eta_k^n]\}\Psi^n(s)]|$$

$$\leq \sum_{j<l<k} |E[\Phi_j^n(\varphi_j^n - \varphi_{j-1}^n)\{\eta_l^n\eta_k^n - E[\eta_l^n\eta_k^n]\}\Psi^n(s)]|$$

$$+ \sum_{l<k} |E[(f'F^n)'F^n(\varphi^n(s))\{\eta_l^n\eta_k^n - E[\eta_l^n\eta_k^n]\}\Psi^n(s)]|$$

($\sum_{j<l<k}$ denotes the summation over (j,l,k)

such that $[ns]+1 \leq j < l < k \leq [nt]$)

$$\leq \sum_{j<l<k} K_0\psi(k-l)^{\frac{1}{2}}\psi(l-j)^{\frac{1}{2}}$$

$$E[|\Phi_j^n(\varphi_j^n - \varphi_{j-1}^n)\Psi^n(s)|]E[|\eta_l^n|]E[|\eta_k^n|]$$

$$+ \sum_{l<k} K_0\psi(k-l)^{\frac{1}{2}}\psi(l-[ns])^{\frac{1}{2}}$$

$$E[|(f'F^n)'F^n(\varphi^n(s))\Psi^n(s)|]E[|\eta_l^n|]E[|\eta_k^n|]$$

$$\leq K_0\left(\sum_{k=1}\psi(k)^{\frac{1}{2}}\right)^2 \{([nt]-[ns])\|\Phi_j^n\|E[|\varphi_j^n - \varphi_{j-1}^n|$$

$$+ \|(f'F^n)'F^n\|\}E[|\eta_l^n|^2]\|\Psi\|$$

$$\to 0 \text{ as } n \to \infty.$$

because of $(III.7)$, $\sup_n n|b^n| < \infty$ and $\sup_n \|G^n\| < \infty$. Hence we get (1).

For (2), (3), (4) and (5), they are easily obtained from Lemma III.1 if we note $E[\eta_k^n] = 0$. For (6), if we think $\sup\|f'\| < \infty$ and $\sup_n \|G^n\| < \infty$ when $n \to \infty$, then we get the result.

For (7), we will first show that for all bounded continuous function g^n on $\mathbb{R}$ such that g^n converge to g,

$$\lim_{n\to\infty} E\left[\sum_{l<k} g^n(\varphi_{l-1}^n)\Psi^n(s)\right] E[\eta_l^n\eta_k^n]$$

$$= E\left[\int_s^t g(\varphi(u))du\Psi(s)\right] V_1 \quad \ldots \quad \circledast$$

where $\Psi(s) = \Psi(\varphi(u_1), \cdots, \varphi(u_m))$ and V_1 is the constant in $(A.II)$ for 1-dimensional case. Set $W_{l,k} = E[\eta_l^n \eta_k^n]$ for $l < k$, which is equal to $W_{1,k-l+1}$ by the stationarity of $\{X_k^n\}$. Then we have :

$$\begin{aligned}
&\sum_{l<k} g^n(\varphi_{l-1}^n) W_{l,k} \\
&\quad = \sum_{l=[ns]+1}^{[nt]-1} g^n(\varphi_{l-1}^n) \sum_{k=l+1}^{[nt]} W_{1,k} \\
&\quad = \sum_{l=[ns]+1}^{[nt]-1} g^n(\varphi_{l-1}^n) \left\{ \sum_{k=2}^{[nt]-[ns]} W_{1,k} - \sum_{k=[nt]-l+2}^{[nt]-[ns]} W_{1,k} \right\}.
\end{aligned}$$

Hence, note that

$$\limsup_{n\to\infty} \sum_{l=[ns]+1}^{[nt]-1} \left(\sum_{k=[ns]-l+2}^{[nt]-[ns]} |W_{1,k}| \right) = 0.$$

and for $s < t$, by the same method of [1],

$$\lim_{n\to\infty} (n) \sum_{k=2}^{[nt]-[ns]} W_{1,k} = V_1$$

Therefore, since $\varphi^n(u) = \varphi_{l-1}^n$ for $u \in [\frac{l-1}{n}, \frac{l}{n}]$,

$$E\left[\sum_{l=[ns]+1}^{[nt]} g^n(\varphi_{l-1}^n) \sum_{k=2}^{[nt]-[ns]} W_{l,k} \Psi^n(s) \right] \to E\left[\int_s^t g(\varphi(u)) du \Psi(s) \right] V_1$$

which is ⊛. Thus we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{l<k} E[(f'F^n)'F^n(\varphi_{l-1}^n)\Psi^n(s)] E[\eta_l^n \eta_k^n] \\
&\quad = E\left[\int_s^t (f'F)'F(\varphi(u)) du \Psi(s) \right] V_1.
\end{aligned}$$

From Lemma III.3, we arrive at the conclusion for $S_1(n)$ of $(III.15)$;

(III.17)

$$\lim_{n\to\infty} E\left[\sum_{k=[ns]+1}^{[nt]} f'(\varphi_{k-1}^n)\left\{F^n(\varphi_{k-1}^n)\eta_k^n + \left(\frac{1}{j_n}\right)G^n(\varphi_{k-1}^n)\right\}\Psi^n(s)\right]$$
$$= E\left[\int_s^t (f'F)'\{F(\varphi(u)) + G(\varphi(u))\}du\Psi(s)\right]V_1.$$

As for $S_2(n)$ of $(III.15)$, since $nb^n \to \int_{\gamma\le|z|\le M} z\nu(dz)$, it is easy to see ;

(III.18)

$$\lim_{n\to\infty} E[S_2(n)\Psi^n(s)] = \lim_{n\to\infty} E\left[\sum_{k=[ns]+1}^{[nt]} f'(\varphi_{k-1}^n)F^n(\varphi_{k-1}^n)b^n\Psi^n(s)\right]$$
$$= E\left[\int_s^t \int_{1\le|z|\le M} (f'F)(\varphi(u))zdu\nu(dz)\Psi(s)\right].$$

Finally, we consider $S_3(n)$ of $(III.15)$. To show the convergence of it is an essential part of establishing the jump-diffusion approximation. To this end, we first prepare the following lemma. First, we notice that φ_k^{n**} in $S_3(n)$ is $\mathcal{F}_{1,k}^n$-measurable and $|\varphi_k^{n**}| \le \frac{1}{2}\|f'\|$.

Lemma III.4.

$$\lim_{n\to\infty} E\left[\sum_{k=[ns]+1}^{[nt]} (\varphi_k^{n**})F^n(\varphi_{k-1}^n)^2(\eta_k^n + b^n)^2\Psi^n(s)\right]$$
$$= E\left[\left\{\int_s^t \left(\frac{1}{2}\right)(f''(F)^2)(\varphi(u))duV_0 + \int_s^t\int_{|z|\le M} H(\varphi(u), z)du\nu(dz)\right\}\Psi(s)\right],$$

where

$$H(x,z)=\int_0^1\int_0^1 \alpha f''(x+\alpha\beta F(x)z)d\alpha d\beta(F(x))^2z^2.$$

Proof. First of all, we will see that

$$\begin{aligned}\varphi_k^{n**}F^n(\varphi_{k-1}^n)^2(\eta_k^n+b^n)^2 &= \frac{1}{2}f''(\varphi_{k-1}^n)F^n(\varphi_{k-1}^n)^2(\eta_{k,\delta}^n)^2\\ &+\int_0^1\int_0^1 \alpha f''(\varphi_{k-1}^n+\alpha\beta F^n(\varphi_{k-1}^n)X_{k,M}^{n,(\delta))})d\alpha d\beta\\ &\quad F^n(\varphi_{k-1}^n)^2(X_{k,M}^{n,(\delta)})^2.\end{aligned} \tag{III.19}$$

As for the first term of the right hand side of $(III.19)$, we get

$$\begin{aligned}\limsup_{n\to\infty}\Bigg|E\Bigg[\sum_{k=[ns]+1}^{[nt]}(f''(F^n)^2)(\varphi_{k-1}^n)\{(\eta_{k,\delta}^n)^2\\ -E[(\eta_{k,\delta}^n)^2]\}\Psi^n(s)\Bigg]\Bigg| = 0 \text{ for } \delta\in C(\nu),\end{aligned} \tag{III.20}$$

because of

$$E[\{(\eta_{k,\delta}^n)^2-E[(\eta_{k,\delta}^n)^2]\}]=0.$$

Combining this with the assumption $(II.3)$, we obtain ;

$$\begin{aligned}\lim_{\delta\to0}\limsup_{n\to\infty}\Bigg|E\Bigg[\sum_{k=[ns]+1}^{[nt]}(f''(F^n)^2)(\varphi_{k-1}^n)(\eta_{k,\delta}^n)^2\Psi^n(s)\\ -\int_s^t(f''(F)^2)(\varphi(u))duV_0\Psi(s)\Bigg]\Bigg|=0\end{aligned} \tag{III.21}$$

Next, as for the second term of the right hand side of $(III.19)$, we set

$$H(x,z)=\int_0^1\int_0^1 \alpha f''(x+\alpha\beta F(x)zd\alpha d\beta(F(x))^2z^2.$$

Then for this function H and each $\delta \in C(\nu)$, it holds ;

$$\text{(III.22)} \qquad \limsup_{n\to\infty} \sum_{k=[ns]+1}^{[nt]} |E[\{H(\varphi_{k-1}^n, X_{k,M}^{n,(\delta)}) - E[H(x, X_{k,M}^{n,(\delta)})]|_{x=\varphi_{k-1}^n}\}\Psi^n(s)]| = 0.$$

Similary, by the above approximation, we can see

(III.23)

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=[ns]+1}^{[nt]} & E[E[H(x, X_{k,M}^{n,(\delta)})]|_{x=\varphi_{k-1}^n} \times \Psi^n(s)] \\ &= E\left[\int_s^t \int_{\delta\le|z|\le M} H(\varphi(u), z)du\nu(dz)\Psi(s)\right] \\ &\to E\left[\int_s^t \int_{|z|\le M} H(\varphi(u), z)du\nu(dz)\Psi(s)\right] \text{ as } \delta \in C(\nu) \downarrow 0.\end{aligned}$$

Combining this with $(III.22)$, we obtain

$$\text{(III.24)} \qquad \lim_{\delta\to 0}\limsup_{n\to\infty} \left| E\left[\sum_{k=[ns]+1}^{[nt]} H(\varphi_{k-1}^n, X_{k,M}^{n,(\delta)})\Psi^n(s) - \int_s^t \int_{|z|\le M} H(\varphi(u), z)du\nu(dz)\Psi(s)\right]\right| = 0.$$

Thus, from $(III.20)$ and $(III.24)$, we arrive at the conclusion of Lemma III.4 .

From the fact $\frac{1}{n}G^n(\varphi_{k-1}^n) \to 0$ as $n \to \infty$, we get for $S_3(n)$;

(III.25)

$$\lim_{n\to\infty} E[S_3(n)\Psi^n(s)] = E\left[\left\{\int_s^t \left(\frac{1}{2}\right)(f''(F)^2)(\varphi(u))duV_0 + \int_s^t \int_{|z|\le M} H(\varphi(u), z)du\nu(dz)\right\}\Psi(s)\right].$$

Now we go back to $(III.15)$. Take $n \to \infty$ in it. Then, for $(III.17)$, $(III.18)$ and $(III.25)$, we obtain the final conclusion ;

$$
\begin{aligned}
&E[\{f(\varphi(t)) - f(\varphi(s))\}\Psi(s)] \\
&= E\left[\int_s^t (f'F)'\{F(\varphi(u)) + G(\varphi(u))\}du\Psi(s)\right] V_1 \\
&\quad + E\left[\int_s^t \int_{1\le|z|\le M} (f'F)(\varphi(u))zdu\nu(dz)\Psi(s)\right] \\
&\quad + E\left[\int_s^t \frac{1}{2}(f''F^2)(\varphi(u))duV_0\Psi\right] \\
&\quad + E\left[\int_s^t \int_{|z|\le M} H(\varphi(u), z)du\nu(dz)\Psi(s)\right] \\
&= E\left[\int_s^t \frac{1}{2}F(\varphi(u))^2 V f''(\varphi(u))du\Psi(s)\right] \\
&\quad + E\left[\int_s^t F(\varphi(u))V_1 F'(\varphi(u))f''(\varphi(u))du\Psi(s)\right] \\
&\quad + E\left[\int_s^t (f'F)'G(\varphi(u))duV_1\Psi(s)\right] \\
&\quad + E\left[\int_s^t \int_{|z|\le M} \{f[\varphi(u) + F(\varphi(u))z] - f(\varphi(u))\right. \\
&\qquad\qquad \left. - F(\varphi(u))zI_{\{|z|\le 1\}}(z)f'(\varphi(u))\}\nu(dz)du\Psi(s)\right] \\
&= E[\mathcal{L}f(\varphi(u))\Psi(s)].
\end{aligned}
$$

Thus

$$E[\{f(\varphi(t)) - f(\varphi(u))\}\Psi(s)] = E[\mathcal{L}f(\varphi(u))\Psi(s)].$$

This proves that

$$
\begin{aligned}
&E[\{M_f(t) - M_f(s)\}\Psi(s)] \\
&\quad = E\left[\left\{f(\varphi(t)) - f(\varphi(s)) - \int_s^t \mathcal{L}f(\varphi(u))du\right\}\Psi(s)\right] \\
&\quad = 0
\end{aligned}
$$

for all $s, t \in J(\varphi)^c, m \in \mathbb{N}, u_i \in J(\varphi)^c, 0 \le u_1 \le u_2 \le \cdots \le u_m \le s$ and bounded continuous functions $\Psi\colon \mathbb{R}^m \to \mathbb{R}$. $Q.E.D.$ of Proposition $III.2$.

We can easily that Proposition III.2 implies that $M_f(t)$ of $(III.13)$ are martingales for all bounded function of calss $C^2(\mathbb{R}^{d+e})$. Therefore, it show that $\tilde{P}_{M,N}$ is a solution of martingale problem in the sense of Jacod-Shiryaev [5].

On the other hand, we have the uniqueness of solutions of martingale problem for $\mathcal{L}_{M,L}$, because the corresponding stochastic differential equation ;

$$
\begin{aligned}
\tilde{\varphi}_{M,L}(t) = \tilde{x}_0 &+ \int_0^t \tilde{F}_L(\tilde{\varphi}_{M,L}(u))B^V(du) \\
&+ \int_0^t \{\tilde{G}_L(\tilde{\varphi}_{M,L}(u)) + \tilde{b}_L(\tilde{\varphi}_{M,L}(u))du \\
&+ \int_0^{t+} \int_{|z|\le\gamma} \tilde{F}_L(\tilde{\varphi}_{M,L}(u-))z\tilde{N}_p(dudz) \\
&+ \int_0^t \int_{\gamma\le|z|\le M} \tilde{F}_L(\tilde{\varphi}_{M,L}(u-))zN_p(dudz)
\end{aligned} \tag{III.26}
$$

has the uniquences solution process.

Hence, we conclude that any limit $\tilde{\varphi}_M$ is equal to the law of the solution $\tilde{\varphi}_{M,N}$ of $(III.26)$, and this yields that

$$\tilde{\varphi}^n_{M,L} \xrightarrow{\mathcal{L}} \tilde{\varphi}_{M,L} \text{ in } \mathbb{D}_{d+e} \text{ as } n \to \infty.$$

$III-2$. Removal of Localization and Trancation

We now proceed into the second step : to remove the restriction of localization. For the purpose, we define the truncated process φ^n_M of φ^n for each $n \in \mathbb{N}$ and $M \in C(\nu)$ by

$$\varphi^n_M(t) = \varphi^n_{[nt],M} \text{ for } t \in [0,\infty). \tag{III.27}$$

where

(III.28)
$$\begin{cases} \varphi^n_{k,M} - \varphi^n_{k-1,M} = F^n(\varphi^n_{k-1,M})(X^n_{k,M} - a^n) \\ \qquad\qquad\qquad\qquad + \left(\dfrac{1}{n}\right) G^n(\varphi^n_{k-1,M}), k = 1, 2, \cdots \\ \varphi^n_{0,M} = x_0 \end{cases}$$

We also define the trancated driving noise process X^n_M of X^n by

(III.29)
$$X^n_M(t) = \sum_{k=1}^{[nt]} (X^n_{k,M} - a^n)$$

and set

(III.30)
$$\tilde{\varphi}^n_M(t) = \begin{pmatrix} \varphi^n_M(t) \\ X^n_M(t) \end{pmatrix}.$$

Our aim in this section is to prove the following proposition for the weak convergence of $\{\tilde{\varphi}^n_M\}_n$.

PropositionIII.3. For each $M \in C(\nu)$, it holds ;

(III.31)
$$\tilde{\varphi}^n_M \xrightarrow{\mathcal{L}} \tilde{\varphi}_M \text{ in } \mathbb{D}_{d+e} \text{ as } n \to \infty.$$

where $\tilde{\varphi}_M$ is the unique solution of the stochastic differential equation ;

(III.32)
$$\begin{aligned} \tilde{\varphi}_M(t) = \tilde{x}_0 &+ \int_0^t \tilde{F}(\tilde{\varphi}_M(u)) B^V(du) \\ &+ \int_0^t \{\tilde{G}(\tilde{\varphi}_M(u) + \tilde{b}(\tilde{\varphi}_M(u))\} du \\ &+ \int_0^t \int_{|z|\le 1} \tilde{F}(\tilde{\varphi}_M(u-)) z \tilde{N}_p(dudz) \\ &+ \int_0^t \int_{1\le |z| \le M} \tilde{F}(\tilde{\varphi}_M(u-)) z N_p(dudz) \end{aligned}$$

We will omit the proof, becase we can get the result by the same method of [1]. By the uniqueness of solution process of $(III.32)$, we can get the uniqueness of martingale problem for

(III.33)
$$\begin{aligned}\tilde{\mathcal{L}}_M f(\tilde{x}) = & \frac{1}{2}\sum_{i,j=1}^{d+e} \left(\tilde{F}(\tilde{x})V\tilde{F}(\tilde{x})^*\right)^{i,j} \frac{\partial^2 f}{\partial \tilde{x}^i \partial \tilde{x}^j}(\tilde{x}) \\ & + \sum_{j=1}^{d+e}\left((\tilde{G}^j(\tilde{x})\frac{\partial f}{\partial \tilde{x}^j}(\tilde{x}) + \tilde{b}^j(\tilde{x})\frac{\partial f}{\partial \tilde{x}^j}(\tilde{x})\right) \\ & + \int_{|z|\le M}\Bigg\{ f(\tilde{x}+\tilde{F}(\tilde{x})z) - F(\tilde{x}) \\ & \qquad - \sum_{j=1}^{d+e}\left(\tilde{F}(\tilde{x})z\right)^j I_{\{|z|\le 1\}}\frac{\partial f}{\partial \tilde{x}^j}(\tilde{x})\Bigg\}\nu(dz).\end{aligned}$$

To remove the trancation in Proposition $III.3$, we think the following ;

For the sequence $\{\tilde{\varphi}_M\}_M$ define by $(III.32)$, it holds

(III.34)
$$\tilde{\varphi}_M \xrightarrow{\mathcal{L}} \tilde{\varphi} \text{ in } \mathbb{D}_{d+e} \text{ as } M \in C(\nu) \to \infty.$$

where $\tilde{\varphi}$ is the process defined by $(II.12)$

To show the tightness of $\{\tilde{\varphi}_M\}_M$, it is suffice to check that for each $f \in C_b^2(\mathbb{R}^{d+e})$, there exists a constant C_f dependinf only on $\|f\| + \|f^{(1)}\| + \|f^{(2)}\|$ such that $f(\alpha(t)) - f(\alpha(0)) - C_f t$ is a supermartingale with respect to $\tilde{p}_M$. It is possible if we take some C_f.

Next, by Proposition III.3 , we have

(III.35)
$$\begin{aligned} E[\{f(\tilde{\varphi}_M(t)) - f(\tilde{\varphi}_M(s)) - \int_s^t \tilde{\mathcal{L}}_M f(\tilde{\varphi}_M(u))du\} \\ \Psi(\tilde{\varphi}_M(u_1),\cdots,\Psi(\tilde{\varphi}_M(u_m))] = 0\end{aligned}$$

for all $s,t \in J(\tilde{\varphi})^c, s < t, m \in \mathbb{N}, u_i \in J(\tilde{\varphi})^c, 0 \le u_1 \le \cdots, \le u_m \le s$, and bounded continuous function $\Psi\colon (\mathbb{R}^{d+e})^m \to \mathbb{R}$. Take $M \to$

∞, then any weak limiting measure of $\{\tilde{\varphi}_M\}_M$ is a solution of the martingale problem for

$$
\begin{aligned}
\tilde{\mathcal{L}}_M f(\tilde{x}) = & \frac{1}{2}\sum_{i,j=1}^{d+e} (\tilde{F}(\tilde{x})V\tilde{F}(\tilde{x})^*)^{i,j}\frac{\partial^2 f}{\partial\tilde{x}^i\partial\tilde{x}^j}(\tilde{x}) \\
& + \sum_{j=1}^{d+e}\left((\tilde{G}^j(\tilde{x})\frac{\partial f}{\partial\tilde{x}^j}(\tilde{x}) + \tilde{b}^j(\tilde{x})\frac{\partial f}{\partial\tilde{x}^j}(\tilde{x})\right) \\
& + \int_{\mathbb{R}^e\setminus\{0\}} \Bigg\{ f(\tilde{x}+\tilde{F}(\tilde{x})z) - f(\tilde{x}) \\
& \qquad - \sum_{j=1}^{d+e}(\tilde{F}(\tilde{x})z)^j I_{\{|z|\le 1\}}\frac{\partial f}{\partial\tilde{x}^j}(\tilde{x})\Bigg\}\nu(dz).
\end{aligned}
\tag{III.36}
$$

To complete the proof, we need some terminology a little more. But anyway we can get the result

$$
\lim_{n\to\infty} E[\Phi(\tilde{\varphi}_n)] = E^{\tilde{P}}[\Phi]
\tag{III.37}
$$

where $\tilde{P}$ is the law of $\tilde{\varphi}$ of $(II.12)$ by the same way as [1]. Since t is arbitrary, $(III.37)$ holds also for all bounded continuous functions Φ. Thus we have Theorem II.

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