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THE WEAK RADON-NIKODYM SETS IN $C(K)$

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1. Introduction.

The space $(X^*, \sigma(X^*, X))$ can be considered as a subspace of a $(C(K), t_p(D))$ space endowed with the topology $t_p(D)$ of pointwise convergence on a dense subset D of a compact Hausdorff space K. Riddle, Saab and Uhl have studied the weak Radom-Nikodym sets in dual Banach spaces ([3], [4], [5], [6]). In particular, Saab [6] showed that if a *weak**-compact convex subset E of the dual X^* of a Banach space X has the WRNP then for any bounded linear operator T from a Banach space Y into X, $T^*(E)$ has the WRNP in Y^*.

In this paper, we investigate some properties of a $P(D)$-set in the $C(K)$ space, which is a generalization of weak Random-Nikodym sets in dual Banach spaces, and we use these results to generalize the Saab's result [6].

2. Preliminaries.

Throughout this paper, K denotes a compact Hausdorff space and $C(K)$ the Banach space of continuous real-valued functions on K endowed with the supremum norm. If F is a subset of K , we denote by $t_p(F)$ the topology in $C(K)$ of pointwise convergence on F. If D is a dense subset of K, then $t_p(D)$ is a Hausdorff locally convex topology in $C(K)$.

Note that if X is a Banach space and B is a norming subset of B_{X^*}, then the convex hull of B, $D = co(B)$, is a dense subset of the compact space $K = (B_{X^*}, weak^*)$, and so considering X as a subspace of $C(B_{X^*})$ the topology $\sigma(X, B)$ is the one induced by $t_p(D)$. In particular, dealing with a dual Banach space X^* the *weak** topology is obtained when we take $K = B_{X^{**}}$ and $D = B = B_X$.

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DEFINITION 2.1 [1]. Let K be a compact Hausdorff space, D a dense subset of K, and H a uniformly bounded $t_p(D)$-compact subset of $C(K)$. H is called a $P(D)$-set if every sequence $\{d_n\}$ in D has a subsequence $\{d_{n_j}\}$ such that $\{h(d_{n_j})\}$ converges for each $h \in H$.

A closed bounded convex subset H of a Banach space X is said to have the weak Radon-Nikodym property (WRNP) if for every complete probability space $(\Omega,\ \Sigma,\ \mu)$ and every vector measure $m\colon \Sigma \to X$ such that the average range of m

$$AR(m) = \left\{ \frac{m(E)}{\mu(E)};\ E \in \Sigma,\ \mu(E) > 0 \right\}$$

is contained in H, there exists a Pettis integrable function $f\colon \Omega \to X$ such that $m(A) = \int_A f d\mu$ for every A in Σ. Some relevant definitions are given in [1], [2] and [7].

The following theorem is useful to obtaining our results.

THEOREM 2.2 [1]. Let H be a uniformly bounded $t_p(D)$-compact convex subset of $C(K)$. Then H is a $P(D)$-set if and only if H has the WRNP.

3. Main results.

Throughout this section, K_1 and K_2 denote compact Hausdorff spaces and D_1 and D_2 denote dense subsets of K_1 and K_2 respectively. For a continuous mapping $T\colon K_1 \to K_2$, define $T^\wedge\colon C(K_2) \to C(K_1)$ by $T^\wedge f(x) = f(Tx)$ for every $f \in C(K_2)$ and $x \in K_1$.

LEMMA 3.1. Let $T\colon K_1 \to K_2$ be a continuous mapping such that $T(D_1) \subseteq D_2$. If H is a uniformly bounded $t_p(D_2)$ -compact convex subset of $C(K_2)$ then $T^\wedge(H)$ is also a uniformly bounded $t_p(D_1)$-compact convex subset of $C(K_1)$.

Proof. It is clear that $T^\wedge(H)$ is uniformly bounded. Since $T^\wedge$ is linear, $T^\wedge(H)$ is convex. To prove that $T^\wedge(H)$ is $t_p(D_1)$-compact, it suffices to prove that
$T^\wedge\colon C(K_2) \to C(K_1)$ is $t_p(D_2)$ - $t_p(D_1)$-continuous. Let $f \in C(K_2)$ be arbitrary and let $\{f_\alpha\}$ be a net in $C(K_2)$ which converges to f

in $t_p(D_2)$-topology. Then $\lim_\alpha f_\alpha(y) = f(y)$ for each $y \in D_2$. Since $T(D_1) \subseteq D_2$, $\lim_\alpha T^\wedge f_\alpha(x) = \lim_\alpha f_\alpha(Tx) = f(Tx) = T^\wedge f(x)$ for each $x \in D_1$. Therefore $\{T^\wedge f_\alpha\}$ converges to $T^\wedge f$ in $t_p(D_1)$-topology. Hence $T^\wedge : C(K_2) \to C(K_1)$ is $t_p(D_2)$ - $t_p(D_1)$-continuous at f, and so $T^\wedge$ is $t_p(D_2)$ - $t_p(D_1)$-continuous.

THEOREM 3.2. Let $T: K_1 \to K_2$ be a continuous mapping such that $T(D_1) \subseteq D_2$ and H a uniformly bounded $t_p(D_2)$ -compact subset of $C(K_2)$. If H is a $P(D_2)$-set in $C(K_2)$ then $T^\wedge(H)$ is a $P(D_1)$-set in $C(K_1)$.

Proof. Suppose that H is a $P(D_2)$-set in $C(K_2)$. By Lemma 3.1, $T^\wedge(H)$ is a uniformly bounded $t_p(D_1)$ -compact subset of $C(K_1)$. Let $\{d_n\}$ be a sequence in D_1. Then $\{Td_n\}$ is a sequence in D_2 because $T(D_1) \subseteq D_2$. Since H is a $P(D_2)$ -set in $C(K_2)$, $\{Td_n\}$ has a subsequence $\{Td_{n_j}\}$ such that $\{h(Td_{n_j})\}$ converges for each $h \in H$. Hence $\{T^\wedge h(d_{n_j})\}$ converges for each $h \in H$. This implies that $T^\wedge(H)$ is a $P(D_1)$-set in $C(K_1)$.

Combining Theorem 2.2 and Theorem 3.2 we obtain :

COROLLARY 3.3. Let $T: K_1 \to K_2$ be a continuous mapping such that $T(D_1) \subseteq D_2$ and H a uniformly bounded $t_p(D_2)$-compact convex subset of $C(K_2)$. If H has the WRNP in $C(K_2)$ then $T^\wedge(H)$ has the WRNP in $C(K_1)$.

LEMMA 3.4. Suppose that $T: K_1 \to K_2$ is continuous and $C(K_2)$ is equicontinuous. If H is a uniformly bounded $t_p(D_2)$-compact convex subset of $C(K_2)$ then $T^\wedge(H)$ is also a uniformly bounded $t_p(D_1)$-compact convex subset of $C(K_1)$.

Proof. It is clear that $T^\wedge(H)$ is uniformly bounded and convex. To prove that $T^\wedge(H)$ is $t_p(D_1)$-compact, it suffices to prove that $T^\wedge : C(K_2) \to C(K_1)$ is $t_p(D_2)$ - $t_p(D_1)$-continuous. Let $f \in C(K_2)$ be arbitrary and let $\{f_\alpha\}$ be a net in $C(K_2)$ which converges to f in $t_p(D_2)$-topology. Then $\lim_\alpha f_\alpha(y) = f(y)$ for each $y \in D_2$. Since D_2 is dense in K_2, for each $x \in D_1$ there exists a net $\{y_\beta\}$ in D_2 such that $\lim_\beta y_\beta = Tx$. Therefore $\lim_\alpha f_\alpha(y_\beta) = f(y_\beta)$ for each β and $\lim_\beta f(y_\beta) = f(Tx)$. We will show that $\lim_{\alpha} f_\alpha(Tx) = f(Tx)$.

Let $\varepsilon > 0$ be given. Since $\lim_{\beta} f(y_\beta) = f(Tx)$, there exists a β_1 such that $\beta \geq \beta_1$ implies $|f(y_\beta) - f(Tx)| < \frac{\varepsilon}{3}$. Since $\{f_\alpha\}$ is equicontinuous, there exists an open neighborhood U of Tx such that $|f_\alpha(y) - f_\alpha(Tx)| < \frac{\varepsilon}{3}$ for each α and $y \in U$. Since $\lim_{\beta} y_\beta = Tx$, there exists a β_2 such that $\beta \geq \beta_2$ implies $y_\beta \in U$. Hence $\beta \geq \beta_2$ implies $|f_\alpha(y_\beta) - f_\alpha(Tx)| < \frac{\varepsilon}{3}$ for all α. Choose a β_0 such that $\beta_1 \leq \beta_0$ and $\beta_2 \leq \beta_0$. Since $\lim_{\alpha} f_\alpha(y_{\beta_0}) = f(y_{\beta_0})$, there exists an α_0 such that $\alpha \geq \alpha_0$ implies $|f_\alpha(y_{\beta_0}) - f(y_{\beta_0})| < \frac{\varepsilon}{3}$. Therefore

$$\begin{aligned}\alpha \geq \alpha_0 \Rightarrow &|f_\alpha(Tx) - f(Tx)| \\ &\leq |f_\alpha(Tx) - f_\alpha(y_{\beta_0})| + |f_\alpha(y_{\beta_0}) - f(y_{\beta_0})| + |f(y_{\beta_0}) - f(Tx)| \\ &< \varepsilon\end{aligned}$$

This implies that $\lim_{\alpha} f_\alpha(Tx) = f(Tx)$. Thus $\lim_{\alpha} T^\wedge f_\alpha(x) = T^\wedge f(x)$ for each $x \in D_1$, and so $\{T^\wedge f_\alpha\}$ converges to $T^\wedge f$ in $t_p(D_1)$-topology. Therefore $T^\wedge\colon C(K_2) \to C(K_1)$ is $t_p(D_2)$ - $t_p(D_1)$-continuous at $f \in C(K_2)$, and so $T^\wedge$ is $t_p(D_2)$ - $t_p(D_1)$ -continuous.

THEOREM 3.5. Suppose that (K_2, ρ) is a compact metric space, $T\colon K_1 \to K_2$ is a continuous mapping and $C(K_2)$ is equicontinuous. If a uniformly bounded $t_p(D_2)$-compact subset H of $C(K_2)$ is a $P(D_2)$-set in $C(K_2)$ then $T^\wedge(H)$ is a $P(D_1)$-set in $C(K_1)$.

Proof. By Lemma 3.4, $T^\wedge(H)$ is uniformly bounded $t_p(D_1)$ -compact in $C(K_1)$. Let $\{d_n\}$ be a sequence in D_1, and let $\varepsilon > 0$ be given. Since H is uniformly bounded and equicontinuous, H is relatively compact in norm topology by Arzela-Ascoli theorem. Hence there exists a $\delta > 0$ such that $\rho(s,t) < \delta$ implies $\sup_{h \in H} |h(s) - h(t)| < \frac{\varepsilon}{2}$. Since D_2 is dense in K_2, for each $n \in \mathbb{N}$ there exists $e_n \in D_2$ such that $\rho(Td_n, e_n) < \delta$. Since H is a $P(D_2)$-set in $C(K_2)$, $\{e_n\}$ has a subsequence $\{e_{n_j}\}$ such that $\{h(e_{n_j})\}$ converges for each $h \in H$. Let $\lim_{j} h(e_{n_j}) = a_h$ for each $h \in H$. Then for each $h \in H$ there exists $N_h \in \mathbb{N}$ such that $j \geq N_h$ implies $|h(e_{n_j}) - a_h| < \frac{\varepsilon}{2}$. Therefore

$$\begin{aligned} j \geq N_h \Rightarrow & \left|h(Td_{n_j}) - a_h\right| \\ & \leq \left|h(Td_{n_j}) - h(e_{n_j})\right| + \left|h(e_{n_j}) - a_h\right| \\ & < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $\lim_j h(Td_{n_j}) = a_h$ for each $h \in H$. That is, $\lim_j T^\wedge h(d_{n_j}) = a_h$ for each $h \in H$. Therefore $\{T^\wedge h(d_{n_j})\}$ converges for each $h \in H$. This implies that $T^\wedge(H)$ is a $P(D_1)$-set in $C(K_1)$.

Combining Theorem 2.2 and Theorem 3.5 we obtain :

COROLLARY 3.6. Suppose that (K_2, ρ) is a compact metric space, $T\colon K_1 \to K_2$ is a continuous mapping and $C(K_2)$ is equicontinuous. If a uniformly bounded $t_p(D_2)$-compact convex subset H of $C(K_2)$ has the WRNP in $C(K_2)$ then $T^\wedge(H)$ has the WRNP in $C(K_1)$.

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