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AN ITERATIVE ROW-ACTION METHOD FOR MULTICOMMODITY TRANSPORTATION PROBLEMS

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Abstract. The optimization problems with quadratic constraints often appear in various fields such as network flows and computer tomography. In this paper, we propose an algorithm for solving those problems and prove the convergence of the proposed algorithm.

1. Introduction

Consider the multicommodity transportation problem with convex quadratic cost function

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}(x-x^0)^T Q(x-x^0) \\ \text{subject to} \quad & \gamma \le Ax \le \delta \end{aligned} \tag{1.1}$$

where $A=(a_{ij})$ is a given $m \times n$ matrix whose ith row is a_i^T, $x^0 \in R^n$, $r, \delta \in R^m$ are given vectors, Q is a given $n \times n$ symmetric positive-definite matrix and the superscript T denotes transposition. We assume that matrix A does not contain any row of which elements are all zero. The pairs of inequality constraints in problem (1.1) are referred to as interval constraints. Interval constraints often appear in optimization problems that arise in various fields such as network flows and computer tomography.

Recently, various row-action methods [1,2,8], which originate from the classical Hildreth's method[4], have drawn much attention. Those

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methods are particularly useful for large and sparse problems, because they act upon rows of the original coefficient matrix one at a time. They are adaptations of coordinate descent methods such as Gauss-Seidel method or its variants, for solving the dual of a given quadratic programming problem. To obtain the solution of problem (1.1), it will be helpful to consider the dual of problem (1.1).

$$\text{(1.2)} \qquad \begin{aligned} \text{minimize} \quad & \phi(z) \\ \text{subject to} \quad & z \geq 0 \end{aligned}$$

where $\phi : R^{2m} \to R$ is a convex quadratic function defined by

$$\text{(1.3)} \qquad \phi(z) = \frac{1}{2} z^T \hat{A} Q^{-1} \hat{A}^T z + z^T (b - \hat{A} x^0),$$

$\hat{A}$ is the $2m \times n$ matrix

$$\text{(1.4)} \qquad \hat{A} = (a_1, -a_1, a_2, -a_2, \cdots, a_m, -a_m)^T,$$

b is the $2m$-vector

$$\text{(1.5)} \qquad b = (\delta_1, -\gamma_1, \delta_2, -\gamma_2, \cdots, \delta_m, -\gamma_m)^T,$$

and z is the $2m$-vector

$$\text{(1.6)} \qquad z = (z_1^+, z_1^-, z_2^+, z_2^-, \cdots, z_m^+, z_m^-)^T.$$

Note that (z_i^+, z_i^-) is a pair of dual variables associated with the ith pair of the interval constraints of (1.1), i.e., z_i^+ and z_i^- correspond to the constraints $a_i^T x \leq \delta_i$ and $-a_i^T x \leq -\gamma_i$, respectively. By taking into account the special structure of problem (1.1), Herman and Lent[3] extended Hildreth's algorithm to deal with interval constraints directly, thereby economizing the number of dual variables by half [2].

Ryang[7] have recently proposed a method that deal with the interval constraints in a direct manner.

In this paper we propose a method for solving those problems, which may be regard as the application of the Jacobi method to the dual of the original problems. We prove the convergence of the proposed algorithm. In section 2, a row-action method is presented. In section 3, the proposed algorithm is shown to converge to the solution of (1.1).

2. Row-Action Method

In this section, we state an algorithm for solving the interval constraint problem (1.1).

ALGORITHM 2.1.

Initialization : Let $(x^{(0)}, x^{(0)}) := (x^0, 0)$, $k := 0$ *and choose a relaxation parameter* $\omega > 0$.

Iteration k *:*

(i) For $i = 1, \cdots, m$,

$$
\begin{aligned}
&\textit{if } z_i^{+(k)} \geq z_i^{-(k)} \textit{ then}\\
&\qquad c_i^{+(k)} := \min\{z_i^{+(k)}, \omega\Delta_i^{(k)}\},\\
&\qquad c_i^{-(k)} := \min\{z_i^{-(k)}, -\omega\Gamma_i^{(k)} + c_i^{+(k)}\}\\
&\textit{else}\\
&\qquad c_i^{-(k)} := \min\{z_i^{-(k)}, -\omega\Gamma_i^{(k)}\},\\
&\qquad c_i^{+(k)} := \min\{z_i^{+(k)}, \omega\Delta_i^{(k)} + c_i^{-(k)}\}\\
&\textit{endif}\\
&\qquad z_i^{+(k+1)} := z_i^{+(k)} - c_i^{+(k)},\\
&\qquad z_i^{-(k+1)} := z_i^{-(k)} - c_i^{-(k)},
\end{aligned}
$$

where

$$
\Delta_i^{(k)} := \frac{\delta_i - a_i^T x^{(k)}}{\alpha_i},
$$

$$
\Gamma_i^{(k)} := \frac{\gamma_i - a_i^T x^{(k)}}{\alpha_i}.
$$

(ii) Let

$$
x^{(k+1)} := x^{(k)} + Q^{-1} \sum_{i=1}^{m} (c_i^{+(k)} - c_i^{-(k)}) a_i.
$$

where

$$
\alpha_i = a_i^T Q^{-1} a_i, \qquad i = 1, \cdots, m. \tag{2.1}
$$

Note that α_i are all positive, since Q is positive definite and $a_i \neq 0$. Note also that, since $\gamma_i \leq \delta_i$, the following inequalities are always satisfied :

$$\Gamma_i^{(k)} \leq \Delta_i^{(k)}, \qquad i = 1, \cdots, m. \tag{2.2}$$

LEMMA 2.1. *Let $\{x^{(k)}\}$ and $\{z^{(k)}\}$ be generated by Algorithm 2.1. Then for all k, we have*

$$x^{(k)} = x^0 - Q^{-1}\hat{A}^T z^{(k)}, \tag{2.3}$$

$$z^{(k)} \geq 0, \tag{2.4}$$

$$z_i^{+(k)} \cdot z_i^{-(k)} = 0, \qquad i = 1, \cdots, m. \tag{2.5}$$

Proof. (2.3) and (2.4) directly follow from the manner in which $\{x^{(k)}\}$ and $\{z^{(k)}\}$ are updated in the algorithm. We prove (2.5) by induction. For $k = 0$, it trivially holds. For each i, we assume $z_i^{+(k)} \cdot z_i^{-(k)} = 0$ and show that it is also true for $k+1$. Without loss of generality, we may only consider the case where $x_i^{+(k)} \geq x_i^{-(k)}$, because a parallel argument is valid for the opposite case. First note that, when $z_i^{+(k)} \geq z_i^{-(k)}$, (2.4) implies $z_i^{+(k)} \geq 0$ and $z_i^{-(k)} = 0$. Moreover, if $z_i^{+(k)} \geq \omega\Delta_i^{(k)}$ holds, then we have $c_i^{+(k)} = \omega\Delta_i^{(k)}$ and $c_i^{-(k)} = \min\{0, \omega(\Delta_i^{(k)} - \Gamma_i^{(k)})\} = 0$, where the last equality follows from (2.2). Therefore we must have $z_i^{-(k+1)} = 0$. On the other hand, if $z_i^{+(k)} < \omega\Delta_i^{(k)}$ holds, then we have $c_i^{+(k)} = z_i^{+(k)}$, which in turn implies $z_i^{+(k+1)} = 0$. Thus (2.5) is satisfied for $k+1$. □ □

For each i, either $z_i^{+(k)} = 0$ or $z_i^{-(k)} = 0$ must always hold by (2.5). Moreover, we can deduce the following relations :

If $z_i^{+(k)} \geq z_i^{-(k)}$, i.e., $z_i^{+(k)} \geq 0$, $z_i^{-(k)} = 0$, then

$$(c_i^{+(k)}, c_i^{-(k)}) = \begin{cases} (\omega\Delta_i^{(k)}, 0), \text{ if } z_i^{+(k)} \geq \omega\Delta_i^{(k)}, \\ (z_i^{+(k)}, 0), \text{ if } \omega\Delta_i^{(k)} \geq z_i^{+(k)} \geq \omega\Gamma_i^{(k)}, \\ (z_i^{+(k)}, -\omega\Gamma_i^{(k)} + z_i^{+(k)}), \text{ if } \omega\Gamma_i^{(k)} \geq z_i^{+(k)}. \end{cases} \tag{2.6}$$

If $z_i^{+(k)} \le z_i^{-(k)}$, i.e., $z_i^{+(k)} = 0$, $z_i^{-(k)} \ge 0$, then
(2.7)

$$(c_i^{+(k)}, c_i^{-(k)}) = \begin{cases} (0, -\omega\Gamma_i^{(k)}), \text{ if } z_i^{-(k)} \ge -\omega\Gamma_i^{(k)}, \\ (0, z_i^{-(k)}), \text{ if } -\omega\Gamma_i^{(k)} \ge z_i^{-(k)} \ge -\omega\Delta_i^{(k)}, \\ (\omega\Delta_i^{(k)} + z_i^{-(k)}, z_i^{-(k)}), \text{ if } -\omega\Delta_i^{(k)} \ge z_i^{-(k)}. \end{cases}$$

3. Convergence of Algorithm 2.1

In this section, we prove convergence of Algorithm 2.1. First, we consider an algorithm for solving general linear complementarity problems. Then we show that Algorithm 2.1 can be reduced to this algorithm.

Let us consider symmetric linear complementarity problem, which is to find $y \in R^l$ such that

$$Mu + q \ge 0, \quad y \ge 0, \quad y^T(My + q) = 0, \tag{3.1}$$

where M is an $l \times l$ symmetric matrix and q is a vector in R^l. If M is positive semidefinite, then this problem is equivalent to the problem

$$\begin{array}{rl} \text{minimize} & \frac{1}{2}y^T My + q^T y \\ \text{subject to} & y \ge 0. \end{array} \tag{3.2}$$

Mangasarian [5] proposes the following algorithm for problem (3.1).

Algorithm 3.1.
Initialization : Let $y^{(0)} := 0$ *and* $k := 0$.
Iteration k *: Choose an* $l \times l$ *diagonal matrix* $E^{(k)}$ *and an* $l \times l$ *matrix* $K^{(k)}$, *and let*

$$y^{(k+1)} := (y^{(k)} - \omega E^{(k)}(My^{(k)} + q + K^{(k)}(y^{(k+1)} - y^{(k)})))_+, \tag{3.3}$$

where, for any vector y, y_+ *denotes the vector with elements* $(y_+)_i = \max\{0, y_i\}$.

Various choices for $\{E^{(k)}\}$ and $\{K^{(k)}\}$ are possible and each particular choice yields a different algorithm [5]. In the following, we show that Algorithm 2.1 is a particular realization of Algorithm 3.1.

First observe that problem (1.2) can be written as problem (3.2) with $l = 2m$ by setting

$$M = \hat{A}Q^{-1}\hat{A}^T, \tag{3.4}$$

$$q = b - \hat{A}z^0, \tag{3.5}$$

$$y = z. \tag{3.6}$$

Note that the positive definiteness of Q implies that the matrix M defined by (3.4) is positive semidefinite. We will show that Algorithm 2.1 can be reduced to Algorithm 3.1 by choosing matrices $E^{(k)}$ and $K^{(k)}$ appropriately.

Specifically, let $E^{(k)}$ and $K^{(k)}$ be $2m \times 2m$ matrices such that

$$E^{(k)} = \begin{pmatrix} D_1^{-1} & & 0 \\ & \ddots & \\ 0 & & D_m^{-1} \end{pmatrix}, \tag{3.7}$$

$$K^{(k)} = \begin{pmatrix} K_1^{(k)} & & 0 \\ & \ddots & \\ 0 & & K_m^{(k)} \end{pmatrix}, \tag{3.8}$$

where

$$D_i^{-1} = \frac{1}{\alpha_i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.9}$$

$$K_i^{(k)} = \begin{cases} \begin{pmatrix} 0 & 0 \\ -\dfrac{\alpha_i}{\omega} & 0 \end{pmatrix}, & \text{if } z_i^{+(k)} \geq z_i^{-(k)}, \\ \begin{pmatrix} 0 & -\dfrac{\alpha_i}{\omega} \\ 0 & 0 \end{pmatrix}, & \text{otherwise,} \end{cases} \tag{3.10}$$

and α_i are defined by (2.1) for all $i = 1, \cdots, m$. Since matrix $K^{(k)}$ given by (3.8) are block diagonal, the pair $(y_{2i-1}^{(k+1)}, y_{2i}^{(k+1)})$ of variables in problem (3.1), which corresponds to $(z_i^{+(k+1)}, z_i^{-(k+1)})$ in problem (1.2) by (1.6) and (3.6), can be updated separately from each other, that is, in parallel for $i = 1, \cdots, m$ [6].

THEOREM 3.1. *Let M, q and y in problem (3.1) be defined by (3.4)-(3.6). Then the sequence $\{z^{(k)}\}$ generated by Algorithm 3.1 with $E^{(k)}$ and $K^{(k)}$ given by (3.7)-(3.10) is identical with the sequence $\{z^{(k)}\}$ generated by Algorithm 3.1 for problem (1.2).*

Proof. The formula (3.3) may be written componentwise as follows : For $i = 1, \cdots, m$, if $z_i^{+(k)} \geq z_i^{-(k)}$ then

$$z_i^{+(k+1)} := \left(z_i^{+(k)} - \frac{\omega}{\alpha_i}(a_i^T Q^{-1} \hat{A}^T z^{(k)} + \delta_i - a_i^T x^0)\right)_+, \tag{3.11}$$

$$\begin{aligned} z_i^{-(k+1)} := \Big(z_i^{-(k)} - \frac{\omega}{\alpha_i}\Big(& - a_i^T Q^{-1} \hat{A}^T z^{(k)} \\ & -\gamma_i + a_i^T x^0 - \frac{\alpha_i}{\omega}(z_i^{+(k+1)} - z_i^{+(k)})\Big)\Big)_+ \end{aligned} \tag{3.12}$$

otherwise

$$z_i^{-(k+1)} := \left(z_i^{-(k)} - \frac{\omega}{\alpha_i}(-a_i^T Q^{-1} \hat{A}^T z^{(k)} - \gamma_i + a_i^T x^0)\right)_+, \tag{3.13}$$

$$\begin{aligned} z_i^{+(k+1)} := \Big(z_i^{+(k)} - \frac{\omega}{\alpha_i}\Big(& a_i^T Q^{-1} \hat{A}^T z^{(k)} \\ & +\delta_i - a_i^T x^0 - \frac{\alpha_i}{\omega}(z_i^{-(k+1)} - z_i^{-(k)})\Big)\Big)_+ . \end{aligned} \tag{3.14}$$

For simplicity, let

$$\bar{x}^{(k)} := x^0 - Q^{-1} \hat{A}^T z^{(k)}, \tag{3.15}$$

$$\bar{\Delta}_i^{(k)} := \frac{\delta_i - a_i^T \bar{x}^{(k)}}{\alpha_i}, \tag{3.16}$$

$$\bar{\Gamma}_i^{(k)} := \frac{\gamma_i - a_i^T \bar{x}^{(k)}}{\alpha_i}. \tag{3.17}$$

Then (3.11)-(3.14) are rewritten as follows : For $i = 1, \cdots, m$, if $z_i^{+(k)} \geq z_i^{-(k)}$ then

(3.18)
$$\begin{aligned} z_i^{+(k+1)} &= \left(z_i^{+(k)} - \frac{\omega}{\alpha_i}(\delta_i - a_i^T \bar{x}^{(k)}) \right)_+ \\ &= \max\{0, z_i^{+(k)} - \omega\bar{\Delta}_i^{(k)}\} \\ &= z_i^{+(k)} - \min\{z_i^{+(k)} - \omega\bar{\Delta}_i^{(k)}\}, \end{aligned}$$

(3.19)
$$\begin{aligned} z_i^{-(k+1)} &= \left(z_i^{-(k)} - \frac{\omega}{\alpha_i} \left(\gamma_i + a_i^T \bar{x}^{(k)} - \frac{\alpha_i}{\omega} \left(z_i^{+(k+1)} - z_i^{+(k)} \right) \right) \right)_+ \\ &= \max\{0, z_i^{-(k)} + \omega\bar{\Gamma}_i^{(k)} + (z_i^{+(k+1)} - z_i^{+(k)})\} \\ &= z_i^{-(k)} - \min\{z_i^{-(k)}, -\omega\bar{\Gamma}_i^{(k)} - (z_i^{+(k+1)} - z_i^{+(k)})\} \end{aligned}$$

otherwise

(3.20)
$$\begin{aligned} z_i^{-(k+1)} &= \left(z_i^{-(k)} - \frac{\omega}{\alpha_i}(-\gamma_i - a_i^T \bar{x}^{(k)}) \right)_+ \\ &= \max\{0, z_i^{-(k)} - \omega\bar{\Gamma}_i^{(k)}\} \\ &= z_i^{-(k)} - \min\{z_i^{-(k)} - \omega\bar{\Gamma}_i^{(k)}\}, \end{aligned}$$

(3.21)
$$\begin{aligned} z_i^{+(k+1)} &= \left(z_i^{+(k)} - \frac{\omega}{\alpha_i} \left(\delta_i - a_i^T \bar{x}^{(k)} - \frac{\alpha_i}{\omega} \left(z_i^{-(k+1)} - z_i^{-(k)} \right) \right) \right)_+ \\ &= \max\{0, z_i^{+(k)} - \omega\bar{\Delta}_i^{(k)} + (z_i^{-(k+1)} - z_i^{-(k)})\} \\ &= z_i^{+(k)} - \min\{z_i^{+(k)}, -\omega\bar{\Delta}_i^{(k)} - (z_i^{-(k+1)} - z_i^{-(k)})\}. \end{aligned}$$

Besides, define $\bar{c}_i^{+(k)}$ and $\bar{c}_i^{-(k)}$ as follows : For $i = 1, \cdots, m$, if $z_i^{+(k)} \geq z_i^{-(k)}$ then

$$\bar{c}_i^{+(k)} := \min\{z_i^{+(k)}, \omega\bar{\Delta}_i^{(k)}\}, \tag{3.22}$$

$$\bar{c}_i^{-(k)} := \min\{z_i^{-(k)}, -\omega\bar{\Gamma}_i^{(k)} + \bar{c}_i^{+(k)}\} \tag{3.23}$$

otherwise

$$\bar{c}_i^{-(k)} := \min\{z_i^{-(k)}, -\omega\bar{\Gamma}_i^{(k)}\}, \tag{3.24}$$

$$\bar{c}_i^{+(k)} := \min\{z_i^{+(k)}, \omega\bar{\Delta}_i^{(k)} + \bar{c}_i^{-(k)}\}. \tag{3.25}$$

It then follows from (3.18), (3.19), (3.22) and (3.23) that, if $z_i^{+(k)} \geq z_i^{-(k)}$, we have

$$z_i^{+(k+1)} = z_i^{+(k)} - z_i^{+(k)}, \tag{3.26}$$

$$z_i^{-(k+1)} = z_i^{-(k)} - z_i^{-(k)}. \tag{3.27}$$

On the other hand, if $z_i^{+(k)} < z_i^{-(k)}$, then (3.20), (3.21), (3.24) and (3.25) imply that the same relations (3.26) and (3.27) also hold.

Moreover note that

$$\begin{aligned}\bar{x}^{(k+1)} &= x^0 - Q^{-1}\sum_{i=1}^m (z_i^{+(k+1)} - z_i^{-(k+1)})a_i \\ &= x^0 - Q^{-1}\sum_{i=1}^m (z_i^{+(k)} - z_i^{-(k)})a_i + Q^{-1}\sum_{j=1}^m (\bar{c}_i^{+(k)} - \bar{c}_i^{-(k)})a_i \\ &= \bar{x}^{(k)} + Q^{-1}\sum_{i=1}^m (\bar{c}_i^{+(k)} - \bar{c}_i^{-(k)})a_i,\end{aligned}$$

where the first and the third equalities follow from (3.24), while the second follows from (3.26) and (3.27). Since both Algorithms 2.1 and 3.1 start with $z^{(0)} = 0$, we can inductively show that

$$\bar{x}^{(k)} = x^{(k)}, \tag{3.28}$$

$$\bar{\Delta}_i^{(k)} = \Delta_i^{(k)}, \quad \bar{\Gamma}_i^{(k)} = \Gamma_i^{(k)}, \qquad i = 1, \cdots, m, \tag{3.29}$$

$$\bar{c}_i^{+(k)} = c_i^{+(k)}, \quad \bar{c}_i^{-(k)} = c_i^{-(k)}, \qquad i = 1, \cdots, m, \tag{3.30}$$

for all k, where $\{x^{(k)}\}, \{\Delta_i^{(k)}\}, \{\Gamma_i^{(k)}\}, \{c_i^{+(k)}\}$ and $\{c_i^{-(k)}\}$ are the sequences generated by Algorithm 2.1. Thus the sequence $\{z^{(k)}\}$ generated by Algorithms 2.1 and 3.1 are identical. □ □

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