EXAMPLES IN ERGODIC THEORY

Kyewon Koh Park* and Seungseol Park

Abstract. In ergodic theory cutting and stacking constructions have been used to obtain a variety of important examples of transformations on the unit interval. We examine the example constructed by J. von Neumann and Kakutani and then apply the method used in the construction of Chacon's transformation to make examples that are weakly mixing but not mixing.

1. Preliminaries

Our discussion will take place on the unit interval $X = [0, 1)$ with $\mathcal{B}$ its family of Lebesgue measurable sets. All sets and functions discussed will be assumed measurable.

Let T be an invertible transformation X onto X. Given a set B and an integer i, let $T^i(B) = \{T^i(x) : x \in B\}$ and $B^T = \cup_{i=-\infty}^{\infty} T^i(B)$. We refer to B^T the set swept out by B.

A transformation T is measurable if $B \in \mathcal{B}$ implies $T(B) \in \mathcal{B}$ and $T^{-1}(B) \in \mathcal{B}$. All transformations are assumed measurable. Hence $B \in \mathcal{B}$ implies $T^i(B) \in \mathcal{B}$ for each integer i and therefore $B^T \in \mathcal{B}$.

Definition 1.1. A transformation T is nonsingular if $m(B) = 0$ if and only if $m(T(B)) = 0$.

That is, T preserves sets of measure zero.

Definition 1.2. A transformation T is measure preserving if for all $B \in \mathcal{B}$, $m(T(B)) = m(B)$.

Definition 1.3. A measure preserving transformation is ergodic if each set of positive measure sweeps out X.

That is, T is ergodic if $m(B) > 0$ implies $m(B^T) = 1$. A set is T-invariant if $TA = A$ in which case $A^T = A$. It is clear that a transformation is ergodic if and only if invariant sets have either measure zero or one.

Definition 1.4. A transformation T is called a σ-translation if there exist disjoint intervals $I_n, n \in N$ and disjoint intervals $J_n, n \in N$ such that $X = \cup_{n=1}^{\infty} I_n = \cup_{n=1}^{\infty} J_n$, I_n and J_n have the same length and T translates I_n onto J_n, $n \in N$.

Lemma 1.5. *All σ-translations are measure preserving.*

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2. Ladders

The ergodicity of the examples will follow from viewing the construction of the examples via ladders. A ladder L of height h and width w is an ordered set of h disjoint subintervals I_i contained in the unit interval $[0,1)$ such that all h intervals have width w and are left-closed and right-open.

Thus $L = \{I_i : 1 \leq i \leq h\}$ and we can view this as a ladder. We refer to I_i as the ith rung, $1 \leq i \leq h$.

The rung I_1 is the base of L and I_h is the top of L. Since all rungs in L are left-closed, right-open, and have the same length, we can define a map T_L that translate I_{i-1} onto $I_i, 2 \leq i \leq h$. Since I_i is directly above $I_{i-1}, 2 \leq i \leq h$, so T_L simply maps a point to the directly above. Let L^* denote the union of the rungs in L, hence T_L is defined on $L^* - I_h$ and T_L^{-1} is defined on $L^* - I_1$.

Given a transformation T, a ladder L is a T-ladder if $T = T_L$ on $L^* - L_h$. In this case iterates of T move a rung up and down the ladder, hence $I^T = L^*$ if I is a rung in L. In particular, if $L^* = [0,1)$, then a rung sweeps out the whole space.

Suppose we start with a ladder L and the partially defined mapping T_L. If I_i is the ith rung in L, as in Figure 2.1, $\cup_{j=-i+1}^{h-i} T_L^j I_i = L^*$. Thus we can say rungs in L sweep out L^* under the action of T_L. Now we can extend T_L so that bisected rungs of L also sweep out L^*. This is accomplished by cutting L in Figure 2.1 in half by a vertical cut down the middle of L. We then obtain two ladders of length h and width $\frac{w}{2}$ each. Let L_1 be the left half and L_2 be the right, as in Figure 2.1. We assume the rungs in L_1 are right-open and the rungs in L_2 are left-closed. We now stack L_2 on top of L_1 to obtain a new ladder L_3 of height $2h$ and width $\frac{w}{2}$, as in Figure 2.2.

Figure 2.1. Figure 2.2

Note that T_{L_3} extends T_L to map the left half I of the top of L onto the right half J at the base of L, as indicated by the heavy arrows in Figure 2.2. Thus the construction of T_{L_3} extends T_L to I which is half of where T_L was not defined.

The extension of T_L is measure preserving since I and J have the same length. Now L_3, $\cup_{j=-i+1}^{2h-i} T_L^j I^i = L_3^*$, that is, each rung in L_3 sweeps out $L_3^* = L^*$.

The preceding construction of cutting in halves and stacking the right half above the left can be repeated inductively. Thus the construction consists of a sequence of ladders L_n cutting in halves and stacking the right above the left half.

3. von Neumann-Kakutani Transformation

The first ladder L_1 is constructed to guarantee that the two binary intervals of length $\frac{1}{2}$ sweep out. Cut $[0,1)$ in half and define $L_1 = ([0,\frac{1}{2}),[\frac{1}{2},1))$ as in Figure 3.1.

Now L_2 is formed to guarantee that the four binary intervals of length $\frac{1}{4}$ sweep out. Cut L_1 in half and stack the right half above the left to form L_2. In general, denote $T_n = T_{L_n}, n \geq 1$. The T_2 extends T_1 by mapping $[\frac{1}{2},\frac{3}{4})$ onto $[\frac{1}{4},\frac{1}{2})$ which is induced by the heavy arrow in Figure 3.1

The induction step starts with a ladder L_n of height 2^n whose rungs are binary intervals of length 2^{-n} and the top of L_n is $[1-2^{-n},1)$.

Thus L_n guarantees that the binary intervals of length 2^{-n} sweep out. Now L_{n+1} formed to guarantee that the binary intervals of length 2^{-n-1} sweep out. Cut L_n in half and stack the right above the left half to obtain L_{n+1}. If $I_n = [1-2^{-n}, 1-2^{-n-1})$ and $J_n = [2^{-n-1},2^{-n})$, then T_{n+1} extends T_n by mapping I_n onto J_n which is indicated by the heavy arrow in Figure 3.2. Thus $T_{n+1}(I_n) = J_n, n \geq 1$ by induction.

Figure 3.1 Figure 3.2

If $I_0 = [0,\frac{1}{2})$ and $J_0 = [\frac{1}{2},1)$, then $[0,1) = \cup_{n=0}^{\infty} I_n = \cup_{n=0}^{\infty} J_n$ and $T_{n+1}(I_n) = J_n, n \geq 0$. If $x \in [0,1)$, then $x \in I_n$ for some $n \geq 0$ and we define $T(x) = T_{n+1}(x)$. Since T_k extends T_n for $k \geq n$, we have $T_k(x) = T_{n+1}(x), k \geq n, x \in I_n$. Therefore we can write $T(x) = \lim_{n\to\infty} T_n(x), x \in [0,1)$. The transformation T extends $T_n, n \geq 1$, hence L_n is a T-ladder, $n \geq 1$ and $T(I_n) = J_n$, $n \geq 0$. Thus T is a σ-translation.

Theorem 3. *The von Neumann-Kakutani transformation T is measure preserving and ergodic.*

Proof. Since T is σ-translation, T is measure preserving. Before verifying ergodicity for T in the general case, first note $I^T = [0,1)$ if I is a rung in L_n, $n \geq 1$. Since L_n consists of the 2^n binary intervals of length 2^{-n}, $n \geq 1$, we have $I^T = [0,1)$ if I is a binary interval. Since every interval contains a binary interval, we have $I^T = [0,1)$ if I is any interval.

In general, let $m(B) > 0$ and choose a point $x \in B$ such that the Lebesgue density of B at x is 1. This means that given $\epsilon > 0$ there exists $\delta > 0$ such that if I is any interval with $x \in I$ and $m(I) < \delta$, then $m(B \cap I) > (1-\epsilon)m(I)$. Choose n so large that $2^{-n} < \delta$ and let $h = 2^n$. There is a binary interval I in L_n with

$x \in I$. Suppose I is the rth rung in L_n, then $T^{-r+1}(I), \cdots, T^{h-r}(I)$ are mutually disjoint, we have

$$\begin{aligned} m(B^T) &\geq m((B \cap I)^T) \\ &= m(\bigcup_{i=-\infty}^{\infty} T^i(B \cap I)) \\ &\geq m(\bigcap_{i=-r+1}^{h-r} T^i(B \cap I)) \\ &= \sum_{i=-r+1}^{h-r} m(T^i(B \cap I)) \\ &= hm(B \cap I) \\ &\geq h(1-\epsilon)m(I) = 1 - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $m(B^T) = 1$. Hence T is ergodic. □

4. Mixing

DEFINITION 4.1. A measure preserving transformation T is mixing if

$$\lim_{n\to\infty} m(T^n(A) \cap B) = m(A)m(B), \quad A, B \in \mathcal{B}.$$

A transformation may not be mixing but can be mixing "on the average".

DEFINITION 4.2. A transformation T is Césaro mixing if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} m(T^i(A) \cap B) = m(A)m(B), \qquad A, B \in \mathcal{B}.$$

LEMMA 4.3. *A transformation T is Césaro mixing if and only if T is ergodic and measure preserving.*

COROLLARY 4.4. *The von Neumann-Kakutani transformation is Césaro mixing but not mixing.*

Proof. It is enough to show that the transformation is not mixing. Let $A = [0, \frac{1}{2})$ and $B = [\frac{1}{2}, 1)$. Note that $T^{2i}(A) \cap B = \emptyset$ for all $i \in Z$. Hence it is not mixing. □

In general, let T be measure preserving and let U_T be the unity operator defined on $\mathcal{L}^2(m)$ by $U_T f(x) = f(T(x))$ for $f \in \mathcal{L}^2(m)$. A complex number c is an eigenvalue for T if there is a corresponding eigenfunction f such that $U_T f = cf$.

REMARK 4.5. *Constant functions are eigenfunctions for any measure preserving transformation with $c = 1$.*

DEFINITION 4.6. A transformation T has continuous spectrum if $c = 1$ is the only eigenvalue for T and constant functions are the only eigenfunctions.

The mixing condition corresponding to continuous spectrum is weakly mixing.

DEFINITION 4.7. A transformation T is weakly mixing if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |m(T^i(A) \cap B) - m(A)m(B)| = 0, \qquad A, B \in \mathcal{B}.$$

A weakly mixing property is difficult to verify directly. The following result of Koopman and von Neumann is generally used to verify weakly mixing.

THEOREM 4.8. *An ergodic measure preserving transformation T is weakly mixing if and only if T has continuous spectrum. [H]*

REMARK 4.9. *The von Neumann-Kakutani transformation is not weakly mixing.*

It is clear that mixing implies weakly mixing and weakly mixing implies Césaro mixing.

5. Examples of weakly mixing transformations

Consider a transformation T with a ladder L of height h. We cut L into four ladders of same width and add an extra interval E above top of the second ladder. The second ladder with E will be stacked above the first ladder, the third ladder will be stacked above the second ladder with E, and the fourth ladder will be stacked above the third ladder. The resulting ladder will have height $4h + 1$, as in Figure 5.1.

Figure 5.1

Example 5.1 Let $a_0 = 0$ and

$$a_n = \sum_{i=1}^{n} \frac{3}{4^{i+1}}, \qquad n \geq 1.$$

Let $E_n = [\frac{3}{4} + a_{n-1}, \frac{3}{4} + a_n), n \geq 1$. The sum of the lengths $\frac{3}{4^{n+1}}$ of E_n, $n \geq 1$, is $\frac{1}{4}$; hence $\cup_{i=1}^{\infty} E_n = [\frac{3}{4}, 1)$.

Let L_1 consist of a single rung $[0, \frac{3}{4})$. This rung is cut into four equal subintervals and E_1 is placed above the second ladder.

We map $[0, \frac{3}{4^2})$ onto $[\frac{3}{4^2}, \frac{6}{4^2}), [\frac{3}{4^2}, \frac{6}{4^2})$ onto E_1, E_1 onto $[\frac{6}{4^2}, \frac{9}{4^2})$, and $[\frac{6}{4^2}, \frac{9}{4^2})$ onto $[\frac{9}{4^2}, \frac{3}{4})$. This results is the ladder L_2.

The base of L_n is $[0, \frac{3}{4^n})$ and the top of L_n is $[\frac{3}{4} - \frac{3}{4^n}, \frac{3}{4})$. Cut L_n into four ladders of width $\frac{3}{4^{n+1}}$ each. Stack the second ladder above the first ladder. Place E_n above the second ladder. Stack the third ladder above the second ladder with E_n and the fourth ladder above the third ladder. The resulting ladder L_{n+1} has width $\frac{3}{4^{n+1}}$ and height $h_{n+1} = 4h_n + 1$. The stacking is equivalent to mapping the top of the first ladder onto the base of the second ladder, the top of the second ladder onto E_n, E_n onto the base of the third ladder, and the top of the third ladder onto the base of the fourth ladder.

If $x \in [0, 1)$, then $x \in [0, \frac{3}{4})$ or $x \in [\frac{3}{4}, 1)$. If $x \in [\frac{3}{4}, 1)$, then $x \in E_n$ for some n and $T_{n+1}(x)$ is defined. If $x \in [0, \frac{3}{4})$, then x is not top of L_n for a sufficiently large n. Hence $T_n(x)$ is well defined. Since T_k extends T_n for $k > n$, we define $T(x) = \lim_{n\to\infty} T_n(x)$, $x \in [0, 1)$.

THEOREM 5.1.1. *T is measure preserving and ergodic.*

Proof. The transformation T is a σ-translation and is therefore measure preserving.

We will prove that T is ergodic. Let B be a set of positive measure and a subset of $[0, 1)$. Choose a point $x \in B$ such that the Lebesgue density of B at x is 1.

Given $\epsilon > 0$ there exists $\delta > 0$ such that if I is any interval with $x \in I$ and $m(I) < \delta$, then $m(B \cap I) > (1 - \epsilon)m(I)$. Choose n so large that $\frac{3}{4^n} < \delta$ and let $h = h_n$. There is an interval I in L_n with $x \in I$. Suppose I is the rth rung in L_n, we have $m(T^i(B \cap I)) \geq (1 - \epsilon)m(I)$, as long as $T^i(I)$ is a rung in L_n. Therefore, we have

$$\begin{aligned}
m(B^T) &\geq m((B \cap I)^T) \\
&\geq m(\bigcap_{i=-r+1}^{h-r} T^i(B \cap I)) \\
&= \sum_{i=-r+1}^{h-r} m(T^i(B \cap I)) \\
&= (1 - \epsilon)m(I)h \\
&= (1 - \epsilon)\frac{3}{4^n}h_n \\
&> (1 - \epsilon)(1 - 4^{-n}).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $m(B^T) = 1$. Hence T is ergodic. □

THEOREM 5.1.2. *T is not mixing.*

Proof. Choosing $A = [0, \frac{3}{4^2})$, it appears as a rung in L_2 and A will be a union of rungs I in L_n for $n \geq 2$. Let I_1, I_2, I_3 and I_4 be the subintervals of equal length of I, respectively. Then $T^h(I_1) = I_2$ and $T^h(I_3) = I_4$. Hence we have

$$m(T^h(I) \cap I) \geq m(T^h(I_1 \cup I_3) \cap I) \geq \frac{m(I)}{2}.$$

Since A is a union of rungs in L and $m(A) < \frac{1}{2}$, it follows that $m(T^h(A) \cap A) \geq \frac{m(A)}{2} > m^2(A)$, and $h = h_n \to \infty$. It follows that T can not be mixing. □

THEOREM 5.1.3. *T is weakly mixing.*

Proof. We want to show that T has a continuous spectrum. Suppose there are f and c such that $f(T(x)) = cf(x)$. If $c = 1$, then $f(T(x)) = f(x)$, that is, f is invariant function under T. Thus f is a constant. It is sufficient to show that $c = 1$.

Since T is measure preserving, $\|f\|_2 = \|f(T)\|_2 = |c|\|f\|_2$, hence $|c| = 1$. Therefore $|f(T(x))| = |cf(x)| = |c||f(x)|$, hence $|f|$ is invariant under T. Since T is ergodic, $|f|$ is constant. Thus we can set $c = e^{ia}$, where $0 \leq a < 2\pi$ and $f(x) = e^{i\theta(x)}$ where $\theta(x)$ is measurable. By Lusin's Theorem there exists a closed set F of measure arbitrarily close to 1 such that $\theta(x)$ is uniformly continuous on F. Therefore given $\eta > 0$, there exists corresponding $\delta > 0$ such that $|\theta(x) - \theta(y)| < \eta$ for all points $x, y \in F$ with $|x - y| < \delta$.

Since $m(F) > 0$, we can choose a point $p \in F$ such that F has Lebesgue density one at p.

Let $\epsilon > 0$. We can choose n sufficiently large so that $\frac{3}{4^n} < \delta$ and there exists a rung I in L_n with $p \in I$ and $m(I \cap F) > (1-\epsilon)m(I)$. If ϵ is sufficiently small, then there must exists $x, y, z \in I \cap F$, where x, y, z are as in Figure 5.1 with $L = L_n$.

Let h denote the height of L_n. Hence, we have

$$e^{i\theta(y)} = f(y) = f(T^h(x)) = c^h f(x) = e^{iha}e^{i\theta(x)} \tag{1}$$

$$e^{i\theta(z)} = f(z) = f(T^{h+1}(y)) = c^{h+1} f(y) = e^{i(h+1)a}e^{i\theta(y)} \tag{2}$$

Thus, we have

$$\theta(y) = ha + \theta(x) \tag{3}$$

$$\theta(z) = (h+1)a + \theta(y) \tag{4}$$

Equalities (3) and (4) are $mod 2\pi$. Since $|x - y| < \delta$ and $|z - y| < \delta$, subtracting (3) from (4) yields $|a + \theta(y) - \theta(x)| = |\theta(z) - \theta(y)| < \eta$. By adding $|\theta(x) - \theta(y)|$ on both sides, we have $|a + (\theta(y) - \theta(x)) + (\theta(x) - \theta(y))| \leq |\theta(z) - \theta(y)| + |\theta(x) - \theta(y)|$, and thus

$$|a| \leq |\theta(z) - \theta(y)| + |\theta(x) - \theta(y)| \leq 2\eta.$$

Since $\eta > 0$ is arbitrary, we obtain $a = 0$, hence $c = 1$. Thus T has continuous spectrum and is therefore weakly mixing. □

REMARK 5.1.4. *The above construction is a variation of the well known Chacon's transformation.[C]*

We will consider putting three extra intervals. Consider a transformation T with a ladder L. We cut L into four pieces of equal width and add an extra interval E_1 above the top of the second ladder, E_2 above the top of the third ladder. Placed E_3 above E_2.

The second ladder with E_1 will be stacked above the first ladder, third ladder with E_2 and E_3 will be stacked above E_1 and the fourth ladder will be stacked above E_3.

Example 5.2 Let $a_{0,3} = 0$ and $a_{n,m} = \frac{1}{2}(\sum_{i=1}^{n-1} \frac{3}{4^i} + \frac{m}{4^n}), \quad 1 \le m \le 3, n \ge 1.$
Let extra intervals be

$$E_{n,1} = [\frac{1}{2} + a_{n-1,3}, \frac{1}{2} + a_{n,1}),$$

$$E_{n,2} = [\frac{1}{2} + a_{n,1}, \frac{1}{2} + a_{n,2})$$

and

$$E_{n,3} = [\frac{1}{2} + a_{n,2}, \frac{1}{2} + a_{n,3}), \quad \text{where} \quad n \ge 1.$$

The length of $E_{n,m}$ are $\frac{1}{2}\frac{1}{4^n}$ for each m. The sum of $E_{n,m}$, $n \ge 1$ and $1 \le m \le 3$ is $\frac{1}{2}$, hence $\cup_{n=1}^{\infty} \cup_{m=1}^{3} E_{n,m} = [\frac{1}{2}, 1)$. Let L_1 consist of a single rung $[0, \frac{1}{2})$. This rung is cut into four equal subintervals. The $E_{1,1}$ is placed above the second ladder, $E_{1,2}$ is placed above the third ladder and $E_{1,3}$ is placed above $E_{1,2}$.

This results in the ladder L_2 of width $\frac{1}{8}$ and height 7. We map $[0, \frac{1}{8})$ onto $[\frac{1}{8}, \frac{1}{4})$, $[\frac{1}{8}, \frac{1}{4})$ onto $E_{1,1}$, $E_{1,1}$ onto $[\frac{1}{4}, \frac{3}{8})$, $[\frac{1}{4}, \frac{3}{8})$ onto $E_{1,2}$, $E_{1,2}$ onto $E_{1,3}$ and $E_{1,3}$ onto $[\frac{3}{8}, \frac{1}{2})$.

For the induction step, we start a ladder L_n of height h_n and width $\frac{1}{2}\frac{1}{4^{n-1}}$ each. The base of L_n is $[0, \frac{1}{2}\frac{1}{4^{n-1}})$ and the top of L_n is $[\frac{1}{2} - \frac{1}{2}\frac{1}{4^{n-1}}, \frac{1}{2})$. Cut L_n into four ladders of width $\frac{1}{2}\frac{1}{4^n}$ each. Place $E_{n,1}$ above the second ladder, $E_{n,2}$ above the third ladder and E_{n3} above the E_{n2}. Stack the second ladder with E_{n1} above the first ladder. Stack the third ladder with $E_{n,2}$ and $E_{n,3}$ above the second ladder with E_{n1}. Stack the fourth ladder above the third ladder with $E_{n,2}$ and $E_{n,3}$. This results in the ladder L_{n+1} with width $\frac{1}{2}\frac{1}{4^n}$ and height $h_{n+1} = 4h_n + 3$.

The ladder L_{n+1} map the top of first ladder onto the base of the second ladder, the top of the second ladder onto $E_{n,1}$, $E_{n,1}$ onto the base of the third ladder, the top of the third ladder onto $E_{n,2}$, $E_{n,2}$ onto $E_{n,3}$ and $E_{n,3}$ onto the base of the fourth ladder.

If $x \in [0, 1)$, then $x \in [0, \frac{1}{2})$ or $x \in [\frac{1}{2}, 1)$. If $x \in [\frac{1}{2}, 1)$, then $x \in E_{nm}$ for some n, m and T_{n+1} is defined. If $x \in [0, \frac{1}{2})$, then x is not in top of L_n for n sufficiently large and $T_n(x)$ is defined. We define $T(x) = \lim_{n\to\infty} T_n(x)$, $x \in [0, 1)$.

THEOREM 5.2.1. *The transformation T is measure preserving, ergodic and weakly mixing but not mixing.*

Proof. We can prove just as in Example 5.1. □

Example 5.3 We consider a sequence of ladders $\{L_n\}$ where the number of cuttings at each step is increasing. Let $k = \frac{1}{e-1}$.

Let $a_0 = 0$ and $a_n = \sum_{i=1}^{n} \frac{k}{(i+1)!}, n \geq 1$ and $E_n = [k + a_{n-1}, k + a_n), n \geq 1$.

The sum of the lengths $\frac{k}{(n+1)!}$ of E_n, $n \geq 1$ is $1 - k$, hence $\cup_{n=1}^{\infty} E_n = [k, 1)$.

Let L_1 consist of a single rung $[0, k)$. This rung is cut into two subintervals. Place E_1 on top of the first interval. We map $[0, \frac{k}{2})$ onto E_1 and E_1 onto $[\frac{k}{2}, k)$. This results in the ladder L_2 of height 3 and width $\frac{k}{2}$ each. The ladder L_2 is cut into three equal ladders and E_2 is placed above the middle ladder. We obtain L_3 of height $h_3 = 3h_2 + 1$ and width $\frac{k}{3!}$. The ladder L_3 is cut into four equal ladders and E_3 is placed above the second ladder. Thus we obtain the ladder L_4 with height $h_4 = 4h_3 + 1$ and width $\frac{k}{4!}$.

For the induction step, we start with a ladder L_n height h_n and width $\frac{k}{n!}$. The base of L_n is $[0, \frac{k}{n!})$ and the top of L_n is $[k - \frac{k}{n!}, k)$. Cut L_n into $n+1$ equal ladders of width $\frac{k}{(n+1)!}$ each. Cutting ladder in L_n are denoted by L_{ni}, $1 \leq i \leq n+1$. Place E_n above ladder $L_{n,[\frac{n+1}{2}]}$, where n is odd and if n is even, place E_n above the middle ladder. Stack $L_{n,i+1}$ ladder with extra interval E_n above $L_{n,i}$. We obtain L_{n+1} of height $h_{n+1} = (n+1)h_n + 1$ and width $\frac{k}{(n+1)!}$. This defines a map T_{L_n} and $T(x) = \lim_{n\to\infty} T_{L_n}(x)$.

THEOREM 5.3.1. *The transformation T is measure preserving, ergodic and weakly mixing but not mixing.*

Proof. The transformation T is a σ-translation and is therefore measure preserving. To prove ergodicity and weakly mixing properties, we proceed exactly as in the Example 5.1.

To see that T is not mixing, choose $A = [0, \frac{k}{2})$, the interval A appears as a rung in L_n, $n \geq 2$. A will be a rung I in L_n for $n \geq 2$. Let h_n be the height of L_n. If n is even, then $m(T^{h_n}(I) \cap I) \geq \frac{n-1}{n+1} m(I)$, $n \geq 2$, hence $m(T^{h_n}(I) \cap I) > \frac{m(I)}{3}$. Since A is a union of rungs in L_n, it follows that $m(T^{h_n}(A) \cap A) > \frac{m(A)}{3}$. Since $m(A) < \frac{1}{3}$ and $h = h_n \to \infty$, it follows that T cannot be mixing. □

Example 5.4 We consider that both the numbers of extra intervals and the cutting numbers are increasing.

Let $a_{00} = 0$ and

$$a_{nm} = \sum_{i=1}^{n-1} \frac{2L}{(2+i)!} \frac{i(i+1)}{2} + \frac{2L}{(2+n)!} m, \qquad 1 \leq m \leq \frac{n(n+1)}{2},$$

where $l = \sum_{n=1}^{\infty} \frac{2}{(2+n)!} \frac{n(n+1)}{2}$ and $L = \frac{1}{1+l}$.

We denote the extra intervals by $E_{n,m}$. Let

$$E_{n,1} = [L + a_{n-1,\frac{n(n+1)}{2}}, L + a_{n,1})\text{and}E_{n,m} = [L + a_{n,m-1}, L + a_{n,m}),$$

where $2 \leq m \leq \frac{n(n+1)}{2}$.

The sum of height $\frac{2L}{(2+n)!}$ of $E_{n,m}$ each m, where $n \geq 1$ and $1 \leq m \leq \frac{n(n+1)}{2}$ is $\frac{l}{1+l}$, hence $\cup_{m=1}^{\infty} \cup_{m=1}^{\frac{n(n+1)}{2}} E_{n,m} = [L, 1)$.

The L_1 consists of the single rung $[0, L)$. This rung is cut into three equal subintervals and $E_{1,1}$ is placed above the middle ladder. We map $[0, \frac{L}{3})$ onto $[\frac{L}{3}, \frac{2L}{3})$, $[\frac{L}{3}, \frac{2L}{3})$ onto $E_{1,1}$ and $E_{1,1}$ onto $[\frac{2L}{3}, L)$. This results in L_2. We cut L_2 into four equal interval and E_{21} is placed above the second ladder and E_{22} is placed above the third ladder. Stack $E_{2,3}$ above $E_{2,2}$. We map the top of the first ladder onto the base of the second ladder, the top of the second ladder onto $E_{2,1}$, $E_{2,1}$ onto the base of the third ladder, the top of the third ladder onto $E_{2,2}$, $E_{2,2}$ onto $E_{2,3}$ and $E_{2,3}$ onto the base the fourth ladder. This results in L_3 with height $h_3 = 4h_2 + 3$ and width $\frac{L}{3 \cdot 4}$ each.

For the induction step, We begin a ladder L_n with height $h_n = (n+1)h_{n-1} + \frac{(n-1)n}{n}$, $\quad n \geq 2$ and width $\frac{2L}{(n+1)!}$. The base of L_n is $[0, \frac{2L}{(2+1)!})$. The top of L_n is $[L - \frac{2L}{(n+1)!}, L)$. Cut L_n into $(n+2)$ equal ladders of width $\frac{2L}{(2+n)!}$ each. We denote cutting ladders in L_n by $L_{n,i}$, $1 \leq i \leq n+2$. Let $m_i = 1 + \frac{i(i-1)}{2}$, $1 \leq i \leq n$. Placed $E_{n,m_{i+j-1}}$, $1 \leq j \leq m_{i+1} - m_i$. We stack $L_{n,i}$ with extra intervals above the $L_{n(i-1)}$, $2 \leq i \leq n+2$. This results in the ladder L_{n+1} of width $\frac{2L}{(2+n)!}$ and height $h_{n+1} = (n+2)h_n + \frac{n(n+1)}{2}$. We can define T_{L_n}. Also we define $T(x) = \lim_{n\to\infty} T_{L_n}(x)$.

THEOREM 5.4.1. *The transformation T is measure preserving, ergodic and weakly mixing.*

REMARK 5.4.2. *It has recently been proved that this transformation T is mixing[AF].*

Note the sharp differences between the examples 5.1-5.3 and the example 5.4.

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Department of Mathematics
Ajou University
Suwon 442-749, Korea