A WEAK SOLUTION OF A NONLINEAR BEAM EQUATION

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ABSTRACT. In this paper we investigate the existence of weak solutions of a nonlinear beam equation under Dirichlet boundary condition on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and periodic condition on the variable t, $u_{tt} + u_{xxxx} = p(x,t,u)$. We show that if p satisfies condition $(p_1) - (p_3)$, then the nonlinear beam equation possesses at least one weak solution.

0. Introduction

In this paper we investigate the existence of weak solutions of a nonlinear beam equation 4 under Dirichlet boundary condition on the interval $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ and periodic condition on the variable t

$$u_{tt} + u_{xxxx} = p(x,t,u) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \tag{0.1}$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \tag{0.2}$$

$$u \text{ is } \pi- \text{ periodic in } t \text{ and even in } x, \tag{0.3}$$

where we shall describe the condition on the function p.

In [3, 4], the authors investigate the existence of multiple solutions of a nonlinear suspension bridge equation (0.1) when the function p is consisted of semilinear terms and the multiple $s\phi_{00}$ ($s \in \mathbb{R}$) of the positive eigenfunction. The existence of multiple solutions of a nonlinear suspension bridge equation with semilinearities crossing multiple eigenvalues was shown by a variational reduction method in [4].

Let L be the beam operator, $Lu = u_{tt} + u_{xxxx}$. Let H_0 be the Banach space spanned by eigenfunctions of the beam operator L, with L^2-norm. Then equation (0.1) with (0.2) and (0.3) is equivalent to the equation

$$Lu = p(x,t,u) \qquad \text{in} \qquad H_0. \tag{0.4}$$

In this paper we assume that the function p satisfies the following.

(1) (p_1) $p(x,t,u)$ is even in x and belongs to $C([-\frac{\pi}{2}, \frac{\pi}{2}]) \times \mathbb{R}, \mathbb{R})$.

(2) (p_2) There are constants $a_1, a_2 \geq 0$ such that

$$|p(x,t,\xi)| \leq a_1 + a_2|\xi|^s \text{ for } 0 \leq s \leq 1,$$

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where in case $s = 1$, we further assume on p that the limits

$$\lim_{\xi\to\infty} \frac{p(x,t,\xi)}{\xi} = -b, \lim_{\xi\to-\infty} \frac{p(x,t,\xi)}{\xi} = -c$$

exit and $-1 < b, c < 15$.

(3) (p_3) There is a constant k with $-3 < k < 1$ such that $p(x,t,\xi) - k = o(|\xi|)$ as $\xi \to 0$.

In Section 1, we investigate the property of the Hilbert space H spanned by eigenfunctions of the beam operator L. We also investigate the property of the Hilbert space.

In Section 2, we first show that the functional corresponding to (0.4) is continuous Fréchet differentiable in a Hilbert space (which is a subspace the Hilbert space H_0) and we calculate several estimates for the functional. By using several estimates and the critical point for a C^1-map in a Banach space, we show that equation (0.4) satisfying (p_1)-(p_3) has at least one solution.

1. The Hilbert space spanned by eigenfunctions

In this section we shall describe the Hilbert space spanned by the eigenfunctions of the one-dimensional wave operator L and investigate the property of it.

When u is even in x and periodic in t with period π, the eigenvalue problem for $u(x,t)$,

$$Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \tag{1.1}$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0,$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding normalized eigenfunctions $\phi_{mn},\ \psi_{mn}$ $(m, n \leq 0)$ given by

$$\begin{aligned}
\phi_{0n} &= \frac{\sqrt{2}}{\pi}\cos(2n+1)x & &\text{for} \quad n \geq 0,\\
\phi_{mn} &= \frac{2}{\pi}\cos 2mt\cdot\cos(2n+1)x & &\text{for} \quad m > 0, n \geq 0,\\
\psi_{mn} &= \frac{2}{\pi}\sin 2mt\cdot\cos(2n+1)x & &\text{for} \quad m > 0, n \geq 0.
\end{aligned}$$

Let n be fixed and define

$$\lambda_n^+ = \inf_m\{\lambda_{mn} : \lambda_{mn} > 0\} = 8n^2 + 8n + 1, \tag{1.2}$$

$$\lambda_n^- = \sup_m\{\lambda_{mn} : \lambda_{mn} < 0\} = -8n^2 - 8n - 3. \tag{1.3}$$

Letting $n \longrightarrow \infty$, we obtain that $\lambda_n^+ \longrightarrow +\infty$ and $\lambda_n^- \longrightarrow -\infty$. Hence, it is easy to check that the only eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17.$$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) : u \text{ is even in } x\}.$$

The set of functions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u, in H_0, as

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}),$$

and we define a subspace H of H_0 as follows

$$H = \{u \in H_0 : \sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2) < \infty\}.$$

Then this space is a Banach space with a norm

$$|||u||| = [\sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2)]^{\frac{1}{2}}.$$

We note that 1 belongs to H_0, but does not belong to H. Hence we can see that the space H is a proper subspace of H_0. The following lemma is very important in this paper [cf. 4].

LEMMA 1.1. *Let c be not an eigenvalue of L and let $u \in H_0$. Then we have $(L-c)^{-1}u \in H$.*

LEMMA 1.2. *If p satisfies (p_1)-(p_2), the map $u(x,t) \to p(x,t,u(x,t))$ belongs to $C(H_0, H_0)$.*

Proof. We note that the function $p(x,t,u(x,t))$ is even in x. If $u \in H_0$, then, by (p_2),

$$\begin{aligned}\int_Q |p(x,t,u)|^2 dxdt &\le \int_Q (a_1 + a_2|u|^s)^2 dxdt \\ &\le a_3 \int_Q (1 + |u|^2) dxdt \qquad (1.4)\end{aligned}$$

for $s \le 0$, which shows that $p : H_0 \to H_0$.

To prove the continuity of this map, observe that it is continuous at u if and only if

$$f(x,t,z(x,t)) = p(x,t,z(x,t) + u(x,t)) - p(x,t,u(x,t))$$

is continuous at $z = 0$.

Therefore we can assume that $u = 0$ and $p(x,t,0) = 0$. Let $\epsilon > 0$. We claim there is a $\delta > 0$ such that $\|u\| \le \delta$ implies $\|P(\cdot,u)\| \le \epsilon$. By (p_1) and $p(x,t,0) = 0$, given any $\hat{\epsilon} > 0$, there is a $\hat{\delta} > 0$ such that

$$|p(x,t,\xi)| \le \hat{\epsilon} \quad \text{if} \quad (x,t) \in Q \quad \text{and} \quad |\xi| \le \hat{\delta}.$$

Let $u \in H_0$ with $\|u\| \leq 0$, δ being free for now, and set

$$Q_1 = \{(x,t) \in Q : |\delta| \leq \beta\}.$$

Then we have

$$\int_{Q_1} |p(x,t,u(x,t))|^2 dxdt \leq \hat{\epsilon}^2 |Q_1| \leq \hat{\epsilon}^2 \pi^2, \tag{1.5}$$

where $|Q_1|$ denotes the measure of Q_1.

Choose $\hat{\epsilon}$ so that $\hat{\epsilon}\pi \leq \frac{\epsilon}{2}$. Let $Q_2 = Q - Q_1$. Then as in (1.4)

$$\int_{Q_2} |p(x,t,u(x,t))|^2 dxdt \leq a_3(|Q_3| + \delta^2). \tag{1.6}$$

Moreover

$$\delta^2 \geq \int^{Q_2} |u|^2 dxdt \geq \beta^2 |Q_2| \tag{1.7}$$

or $|Q_2| \leq (\delta\beta^{-1})^2$. Combining (1.6)-(1.7) gives

$$\int_{Q_2} |p(x,t,u)|^2 dxdt \leq a_3(1+\beta^{-2})\delta^2. \tag{1.8}$$

Choose δ so that $a_3(1+\beta^{-2})\delta^2 \leq (\frac{\epsilon}{2})^2$. Then (1.5) and (1.8) imply $\|p(\cdot,u)\| \leq 0$ if $\|u\| \leq \delta$. This completes the lemma. □ □

With above lemmas 1.1 and 1.2, we can obtain the following.

Proposition 1.1. *Assume that the function p satisfies the conditions (p_1)-(p_2). If we have a solution, in H_0, of the equation*

$$Lu = p(x,t,u) \quad in \quad H_0, \tag{1.9}$$

then it belongs to H.

Proof. Assume that u belong to the Hilbert space H_0. Since p satisfies (p_1) and (p_2), it follows from Lemma 1.2 that $p(x,t,u)$ belongs to H_0. Equation (1.9) is equivalent to

$$u = L^{-1}[p(x,t,u)]$$

Hence it follows from Lemma 1.1 that u belongs to H. □ □

With aid of Proposition 1.1, the investigation of the existence of solutions in H_0 of (1.9) reduces to the investigation of one in H of (1.9).

2. The existence of a weak solution

In this section we investigate the existence of a nontrivial solutions of the nonlinear beam equation (1.9) described in Section 1. By Proposition 1.1, problem (1.9) in H_0 is reduced to the one in the subspace H of the Hilbert space H_0. Hence we consider the problem

$$Lu = p(x,t,u(x,t)) \quad \text{in} \quad H, \tag{2.1}$$

where the function p satisfies $(p_1) - (p_3)$ described in Section 0.

We now state the main theorem of this section.

THEOREM 2.1. *Assume that p satisfies* $(p_1)-(p_3)$. *Then (2.1) possesses at least one solution.*

We now consider the functional associated with (2.1)

$$I(u) = \int_Q [\frac{1}{2}(-|u_t|^2 + |u_{xx}|^2) - P(x,t,u)]dxdt, \tag{2.2}$$

where $P(x,t,\xi) = \int_0^\xi p(x,t,\eta)d\eta$.

Since every solution in H_0 of (2.1) belongs to H, it suffices to investigate the solutions in H of (2.1). On the other hand, the weak solutions of (2.1) coincide with the critical points of $I(u)$. Hence we consider the functional $I(u)$ in H.

In order to apply critical point theory to the functional $I(u)$ given by (2.2), we have to know that $I \in C^1(H, \mathbb{R})$ and $I(u)$ satisfies the Palais-Smale condition.

First we prove that I is continuous and Fréchet differentiable in H.

LEMMA 2.1. *If p satisfies* (p_1)-(p_3), *then the functional* $I(u)$ *is continuous and Fréchet differentiable in* H, *and* $I'(u)$ *is continuous in* H *with*

$$I'(u)\phi = \int_Q [Lu\phi - p(x,t,u)\phi]dxdt \tag{2.3}$$

for all $\phi \in H$. *Moreover*

$$J(u) = \int_Q P(x,t,u)dxdt$$

is weakly continuous and $J'(u)$ *is compact.*

Proof. Let u be in H and prove that $I(u)$ is continuous at u. We consider

$$I(u+v) - I(u) = \int_Q [u(v_{tt} + v_{xxxx}) + \frac{1}{2}v(v_{tt} + v_{xxxx})]dxdt$$
$$- \int_Q [P(x,t,u+v) - P(x,t,u)]dxdt.$$

Let $u = \sum h_{mn}\phi_{mn} + k_{mn}\psi_{mn}$, $v = \sum \tilde{h}_{mn}\phi_{mn} + \tilde{k}_{mn}\psi_{mn}$. Then, by using Schwartz inequality, we have

$$|\int_Q u(v_{tt} + v_{xxxx})dxdt| \le \|\|u\|\| \cdot \|\|v\|\|,$$

$$|\int_Q \frac{1}{2}v(v_{tt} + v_{xxxx})dxdt| \le \|\|v\|\|^2.$$

On the other hand, by Mean Value Theorem, we have

$$P(x,t,\xi+\eta) - P(x,t,\xi) = P(x,t,\xi+\theta\eta)\eta \quad \text{for some} \quad \theta \in (0,1).$$

Therefore by (p_2), we have, for $s \leq 1$,

$$
\begin{aligned}
&\int_Q |P(x,t,u+v) - P(x,t,u)|dxdt \\
&= \int_Q |p(x,t,u+\theta v)||v|dxdt \\
&\leq \int_Q [a_1 + a_2(|u| + |v|)^s]|v|dxdt \\
&\leq a_1\|v\| + a_2\|u\|\|v\| + a_2\|v\|^2 \\
&\leq a_1|||v||| + a_2|||u||| \cdot |||v||| + a_3|||v|||^2 \\
&= (a_1 + a_2|||u||| + a_3|||v|||)|||v|||. \qquad (2.4)
\end{aligned}
$$

With the above results, we can see that $I(u)$ is continuous at u.

Now let us prove that $I(u)$ is Fréchet differentiable at $u \in H$ with equation (2.3). To prove equation (2.3), we compute the following.

$$
\begin{aligned}
&|I(u+v) - I(u) - I'(u)v| \\
=&|\int_Q [u(v_{tt} + v_{xxxx}) + \frac{1}{2}v(v_{tt} + v_{xxxx}) - (P(x,t,u+v) - P(x,t,u)) \\
&- (u_{tt} + u_{xxxx})v + p(x,t,u)v]dxdt| \\
=&|\int_Q \frac{1}{2}vLvdxdt - \int_Q [P(x,t,u+v)P(x,t,u) - p(x,t,u)v]dxdt| \\
\leq&\frac{1}{2}|||v|||^2 + \int_Q |p(x,t,u+\theta v) - p(x,t,u)||v|dxdt.
\end{aligned}
$$

By (p_1), $p(x,t,\xi)$ is a continuous function of ξ and hence for given $\epsilon_1 > 0$, there exists $\delta > 0$ such that

$$|p(x,t,u+\theta v) - p(x,t,u)| < \epsilon_1$$

holds almost everywhere when $|||v||| < \delta$. Therefore we have

$$
\begin{aligned}
|I(u+v) - I(u) - I'(u)v| &< \frac{1}{2}|||v|||^2 + \epsilon_1 \int_Q |v|dxdt \\
&\leq \frac{1}{2}|||v||| + \epsilon_1|||v||| \\
&\leq (\frac{1}{2}|||v||| + \epsilon_1)|||v|||,
\end{aligned}
$$

which proves equation (2.3).

It is clear that the first term in I' is continuous. Hence, to prove the continuity of $I'(u)$, it suffices to show that $J'(u)$ is continuous. Let $u_m \to u$ in H. Then $u_m \to u$

in H_0 and we have

$$\begin{aligned}\|J'(u_m) - J'(u)\|_{op} &= \sup_{|||\phi|||\leq 1} |\int_Q (p(x,t,u_m) - p(x,t,u))\phi dxdt| \\ &\leq \sup_{|||\phi|||\leq 1} \int_Q |p(x,t,u_m) - p(x,t,u)||\phi| dxdt \\ &\leq \sup_{|||\phi|||\leq 1} \|p(x,t,u_m)p(x,t,u)\|\|\phi\| \\ &\leq \sup_{|||\phi|||\leq 1} \|p(x,t,u_m) - p(x,t,u)\| \cdot |||\phi||| \\ &\leq \|p(x,t,u_m) - p(x,t,u)\|, \qquad (2.5)\end{aligned}$$

where $\|\cdot\|_{op}$ is the operator norm. Since the map $u(x,t) \to p(x,t,u(x,t))$ belongs to $C(H_0, H_0)$, the last term in the above inequalities tends to 0 as $m \to \infty$ and J' is continuous.

To prove that J is weakly continuous, let u_m converges to u in H. Then u_m converges to u in H_0 since $\|u\| \leq |||u|||$. Consequently, Lemma 1.2 implies $J(u_m) \to J(u)$.

Finally, to prove that J' is compact, let (u_m) be bounded in H. Then along a subsequence, u_m converges weakly to some $u \in H$ and $u_m \to u$ in H_0. The proof then concludes via (2.3). □ □

Lemma 2.2. *If p satisfies (p_1)-(p_3), then the functional $I(u)$ satisfies the Palais-Smale condition. That is, any sequence (u_m) in H for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ in H as $m \to \infty$ possesses a convergent subsequence.*

The verification for (PS) is simplified with aid of the following result.

Lemma 2.3. *Let p satisfies (p_1)-(p_2). If (u_m) is a bounded sequence in H such that $I(u_m)$ is bounded and $I'(u_m) \to 0$ in H as $m \to 0$, then (u_m) has a convergent subsequence.*

Proof. Suppose that $I(u_m)$ is bounded and $I'(u_m) \to 0$ in H as $m \to \infty$ for any bounded sequence (u_m) in H. Let $D : H \to H^*$ denote the duality map between H and its dual defined by

$$(Du)\phi = \int_Q Lu \cdot \phi dxdt \quad \text{for} \quad u, \phi \in H.$$

Thus

$$D^{-1}I'(u) = u - D^{-1}J'(u).$$

By the continuity of D^{-1} and (2.10), we have

$$u_m = D^{-1}I'(u_m) + D^{-1}J'(u_m) \ \to \ D^{-1}J'(u_m),$$

where the limit being taken along the convergent subsequence of $J'(u_m)$.

But, since (u_m) is bounded in H and J' is compact (cf. Lemma 2.1), $J'(u_m)$ has a convergent subsequence. This completes the lemma. □ □

We now state the theorem (cf. [3]), which will be useful in the proof of Lemma 2.2.

THEOREM 2.2. *Let $-1 < a, b < 15$. Then the equation*

$$Lu + au^+ - bu^- = 0 \quad in \quad H_0$$

has only the trivial solution.

Proof of Lemma 2.2. By Lemma 2.3, to verify (PS), we need only show that $|I(u_m)| \le M$ and $I'(u_m) \to 0$ as $m \to \infty$ implies that (u_m) is a bounded sequence. For m large, we have

$$Lu_n + p(x,t,u_n) = DI(u_n) \quad \text{in} \quad H.$$

Assume that (PS) condition does not hold, that is, $|||u_n||| \to \infty$. Dividing by $|||u_n|||$ and taking $w_n = |||u_n|||^{-1}u_n$, we have

$$Lw_n + \frac{1}{|||u|||}p(x,t,u_n) = \frac{1}{|||u_n|||}DI(u_n). \tag{2.6}$$

Since $DI(u_n) \to 0$ as $n \to \infty$ and $|||u_n||| \to \infty$, the right hand side of (2.6) converges to 0 in H as $n \to \infty$. Moreover (2.6) shows that $|||u_n|||$ is bounded. Since L^{-1} is a compact operator, passing to a subsequence we get that $w_n \to w_0$ in H. Since $|||w_n||| = 1$ for all $n = 1, 2, \cdots$, it follows that $|||w_0||| = 1$. Taking the limit of (2.6), we find

$$Lw_0 + bw_0^+ - cw_0^- = 0 \text{ (in case } s = 1), \text{ or } Lw_0 = 0 \text{ (in case } 0 \le s < 1)$$

with $|||w_0||| \ne 0$. This contradicts to the fact (from Theorem 2.2) that the above equation has only the trivial solution. □ □

We state a critical point theory, which is very useful to show the existence of critical points of a C^1-map in a Banach space.

LEMMA 2.4. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying (PS). Then the local minimum (or maximum) of I is a critical point of I.*

LEMMA 2.5. *Suppose that I satisfy $(p_1) - (p_3)$. Then, as $u \to 0$, we have*

$$I(u) = \frac{1}{2}\int_Q Lu \cdot u dx dt + \int_Q ku dx dt + o(|||u|||^2). \tag{2.7}$$

Proof. By (p_3), given any $\epsilon > 0$, there is a $\delta > 0$ such that $|\xi| \le \delta$ implies

$$|P(x,t,\xi) - k\xi| \le \frac{1}{2}\epsilon|\xi|^2$$

for all $(x,t) \in Q$. By (p_2), there is a constant $A = A(\delta) > 0$ such that $|\xi| > \delta$ implies

$$|P(x,t,\xi) - k\xi| \le A|\xi|^{s+1}$$

for all $(x,t) \in Q$. Combining these two estimates, for all $\xi \in \mathbb{R}$ and $(x,t) \in Q$,

$$|P(x,t,\xi) - k\xi| \leq \frac{\epsilon}{2}|\xi|^2 + A|\xi|^{s+1}.$$

On the other hand, there is a δ_1 such that $|\|u\|| < \delta_1$ implies

$$\int_{|u(x,t)|>\delta} |P(x,t,u) - ku|dxdt \leq \frac{\epsilon}{2}|\|u\||.$$

Therefore if $|\|u\|| < \delta_1$, we have

$$\begin{aligned} |J(u) - k\int_Q udxdt| &\leq \frac{\epsilon}{2}\int_{|u(x,t)|\leq\delta} |u|^2 dxdt + A\int_{|u(x,t)|>\delta} |u|^{s+1}dxdt \\ &\leq \frac{\epsilon}{2}\|u\|^2 + \frac{\epsilon}{2}|\|u\|| \\ &\leq \epsilon|\|u\||^2 \end{aligned}$$

for $s \leq 1$. Hence

$$|J(u) - k\int_Q udxdt| \leq \epsilon|\|u\||^2.$$

Since ϵ was arbitrary, $J(u) - k\int_Q udxdt = o(|\|u\||^2)$ as $u \to 0$. Therefore we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_Q Lu \cdot udxdt - J(u) \\ &= \frac{1}{2}\int_Q Lu \cdot udxdt + \int_Q kudxdt + o(|\|u\||^2).\square \end{aligned}$$

□

Let V be the subspace of H, spanned by the eigenfunctions of $\lambda_{mn} > 0$ and W be the orthogonal compliment of V in H. Let $P : H \to V$ denote the orthogonal projection of H onto V and $I - P : H \to W$ that of H onto W. Then every element u of H is expressed by $u = v + w$, where $v = Pu$, $w = (I - P)u$. Hence equation (2.1) is equivalent to a system

$$Lv = P(p(\cdot,\cdot,v+w)), \qquad (2.8.a)$$

$$Lw = (I-P)(p(\cdot,\cdot,v+w)). \qquad (2.8.b)$$

We let

$$I_1(v) = \frac{1}{2}\int_Q Lv \cdot vdxdt - \int_Q P(x,t,v+w)dxdt,$$

$$I_2(w) = \frac{1}{2}\int_Q Lw \cdot wdxdt - \int_Q P(x,t,v+w)dxdt.$$

Then we have the following lemma.

LEMMA 2.6. *There is a neighborhood B_1 of 0 in V such that for any $v \in B_1$ there exists a solution $z \in W$ of equation (2.8.b) in W, where z is a local maximum $I_2(w)$. If we put $z = \theta(v)$ in B_1, then θ is continuous in B_1 and we have*

$$DI(v + \theta(v))(w) = 0 \quad \textit{for all} \quad w \in W.$$

Proof. Equation (2.8.b) is equivalent to

$$z = L^{-1}(I - P)(p(\cdot, \cdot, v + w)). \tag{2.9}$$

We note that $L^{-1}(I - P)$ is a self-adjoint, compact, linear map from W into itself and the eigenvalues $L^{-1}(I - P)$ in W are λ_{mn}^{-1} with $\lambda_{mn} < 0$. Hence, by Lemma 2.5,

$$I_2(w) = -\frac{1}{2}|||w|||^2 + \int_Q k(v + w)dxdt + o(|||v|||^2)o(|||w|||^2).$$

Since $-3 < k < 1$, $I_2(w)$ has a local maximum z, which is a solution of (2.8.b). We let $z = \theta(v)$. By Lemma 1.2 and Lemma 2.2 of [3], θ is continuous in B_1.

Let $v \in V$ and set $z = \theta(v)$. If $w \in W$, then from (2.2) we see that

$$\int_Q (-z_t w_t + z_x w_x + p(x, t, v + w)w)dxdt = 0.$$

Since $\int_Q v_t w_t = 0$ and $\int_Q v_x w_x = 0$, we have

$$DI(v + \theta(v))(w) = 0 \quad \text{for} \quad w \in W. \square$$

$\square$

Proof of Theorem 2.1. It follows from Lemma 2.2 of [3] that if $\tilde{I} : V \to \mathbb{R}$ is defined by $\tilde{I}(v) = I(v + \theta(v))$ in B_1, then $\tilde{I}$ has a continuous Fréchet derivative $D\tilde{I}$ with respect to v and

$$D\tilde{I}(v)(h) = DI(v + \theta(v))(h) \quad \text{for all} \quad h \in V.$$

If v_0 is a critical point of $\tilde{I}$, then $v_0 + \theta(v_0)$ is a solution of (2.1). Since V is the subspace of H of eigenfunctions of $\lambda_{mn} > 0$, $\tilde{I}$ has a local minimum in B_1, which is a critical value of $\tilde{I}$. This completes the proof of Theorem 2.1. $\square$ $\square$

References

[1] H. Amann, *Saddle points and multiple solutions of differential equations*, Math. Z. (1979), 127–166.

[2] A. Ambrosetti and P.H. Rabinobitz, *Dual variational methods in critical point theory and applications*, J. Funt. Anal. **14** (1973), 349–381.

[3] Q-H. Choi and T. Jung, *On periodic solutions of the nonlinear suspension bridge equation*, Diff. Int. Equations **4** (1991), 383–396.

[4] Q-H. Choi, T. Jung, and P.J. McKenna, *The study of a nonlinear suspension bridge equation by a variational reduction method*, Applicable Analysis **50** (1993), 73–92.

[5] A.C. Lazer and P.J. McKenna, *Critical points theory and boundary value problems with nonlinearities crossing multiple eigenvalues II*, Comm. Partial Differential Equations **11** (1986), 1653–1676.

[6] P.J. McKenna and W. Walter, *Nonlinear oscillations in a suspension bridge*, Archive for Rational Mechanics and Analysis **98** (1987), 167–177.

[7] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, Conference board of the mathematical sciences regional conference series in mathematics, vol. 64, Amer. Math. Soc., Providence, RI, 1988.

[8] M.Struwe, *Variational Methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, Springer-Verlag, Berlin, 1990.

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