A NOTE ON HARMONIC MAPPINGS

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Abstract. In this note, we study a relation between harmonic maps and exponential harmonic maps, and we show existence of Yang-Mills connections.

1. Introduction

The theory of harmonic maps and exponentially harmonic maps has recently developed very much as we see excellent expository papers of Eells and Lamaire [2].

In this paper we focus on the exponentially harmonic maps and exponentially Yang–Mills connections. It is well known that both theories of harmonic maps and Yang–Mills connections have certain strong similarities. We introduce and study another problem of calculus of variations in an analogous way as exponentially harmonic maps. Namely, we define the exponential Yang–Mills connection and we show existence of Yang–Mills connection and exponential Yang–Mills connection.

2. Main Results

Let (M, g) and (N, h) be two compact Riemannian manifolds and $\varphi : M \to N$ be a smooth map. Harmonic maps are extremals of the energy functional

$$E(\varphi) = \int_M e(\varphi) v_g$$

where $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density and v_g is the canonical volume element. The map φ is harmonic if and only if it satisfies the Euler–Lagrange equation

$$\tau(\varphi) = div(d\varphi) = 0.$$

The existence problem for harmonic maps is the following; Given two Riemannian manifolds (M, g), (N, h) and a homotopy class $\mathcal{H}$ of smooth maps from M to N, when is there a harmonic maps in $\mathcal{H}$? This problem has been studied extensively, and the answer depends on the manifolds and the homotopy class. To obtain existence of solutions in all dimensions without conditions on the manifolds, Eells–Lemaire [2] considered another problem of calculus of variations. They defined the exponential energy of φ as

$$E_e(\varphi) = \int_M \exp(\frac{1}{2}|d\varphi|^2) v_g,$$

and say that a smooth extremal of E_e an exponentially harmonic maps.

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Proposition 1. [2] *Let (M,g) and (N,h) be compact manifolds, $\mathcal{H}$ a homotopy class. Then $\mathcal{H}$ contains an $\mathbb{E}$–minimizing map, which is α–Hölder continuous for all $\alpha < 1$.*

This can be verified using the properties of the Sobolev spaces of maps from M to N, which are defined as follows. Choose a finite atlas on M and Riemannian embedding of (N,h) in some Euclidean space V. Let $\mathcal{L}_1^p(M,V)$ be the Sobolev space of L^p functions from M to V whose first partial derivatives are also L^p. Then we set

$$\mathcal{L}_1^p(M,N) = \{\phi \in \mathcal{L}_1^p(M,V),\ \phi(x) \in N\ a.e.\}$$
$$W = \cap_{p\geq 1}\mathcal{L}_1^p(M,N),$$

and consider in $W \cap \mathcal{H}$ a minimizing sequence (ϕ_n) for $\mathbb{E}$. Since

$$\mathbb{E}(\phi) = \int_M \sum_{k=0}^{\infty} \frac{1}{k!}(\frac{|d\phi|^2}{2})^k v_g,$$

(ϕ_n) is bounded in each $\mathcal{L}_1^p(M,N)$. Using the compactness of various Sobolev embeddings and a diagonal argument, we deduce that a subsequence converge weakly in each $\mathcal{L}_1^p$, strongly in each $\mathcal{L}^p$, and in C^α for each $\alpha < 1$. In particular, the convergence is uniform and the limit ϕ belongs to the homotopy class $\mathcal{H}$. Convexity in P of $\exp(|p|^2/2)$ insure lower–semicontinuity of $\mathbb{E}$ for that convergence, so that $\mathbb{E}(\phi) < \liminf\,(\phi_n)$, and ϕ is a C^α minimizer.

Theorem 2. [1] *If $\dim M \geq 3$ for any homotopy class $\mathcal{H}$, then there exist a C^∞ Riemannian metric $\tilde{g}$ on M conformal to g and C^∞ map φ in $\mathcal{H}$ such that $\varphi : (M,\tilde{g}) \to (N,h)$ is harmonic.*

Theorem 3. [3] *If $\dim M \geq 3$, then there is a smooth metric $\tilde{g}$ conformally equivalent to g and a map $\varphi \in \mathcal{H}$ such that $\varphi : (M,\tilde{g}) \to (N,h)$ is exponentially harmonic.*

We prove a relation between exponentially harmonic and harmonic maps.

Theorem 4. *If $\dim M \geq 3$ and $\varphi : (M,\tilde{g}) \to (N,h)$ is exponentially harmonic, then there exists a smooth metric $\tilde{g}$ conformally equivalent to g such that $\varphi : (M,\tilde{g}) \to (N,h)$ is harmonic.*

Corollary. *Let $\varphi : (M,g) \to (N,h)$ be an exponentially harmonic map which is constant on an open subset of M. Then φ is constant on M.*

It is well–known that both theories of Yang–Mills connections and harmonic maps have certain similarities. We introduce and study another problem of calculus of variations in an analogous way as exponentially harmonic maps. Namely, we define the exponential Yang–Mills functional. Let (M,g) be a compact Riemannian manifold, and let E be a G–vector bundle over M. Let $C(E)$ be the space of all C^∞ G–connections of E. For $\nabla \in C(E)$, let R^∇ be its curvature tensor. The Yang–Mills functional $yM : C(E) \to R$ is defined by

$$yM(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2 v_g.$$

DEFINITION. The exponential Yang-Mills functional $yM_e : C(E) \to R$ is defined by

$$yM_e(\nabla) = \int_M \exp(\frac{1}{2}\|R^\nabla\|^2)v_g.$$

A critical point $\nabla \in C(E)$ of the Yang–Mills functional yM is called a Yang–Mills connection and a critical point of the exponential Yang–Mills functional yM_e is called an exponential Yang–Mills connection.

THEOREM 5. *Let (M, g) be an n–dimensional Riemannian manifold, G a compact Lie group and E a G–vector bundle over M. Assume that $n \geq 5$. Then there exists a C^∞ Riemannian metric $\tilde{g}$ on M conformal to g and a C^∞ G–connection ∇ on E such that ∇ is Yang–Mills connection with respect to $\tilde{g}$.*

Proof. For a positive C^∞–function f on M, put a new Riemannian metric $\tilde{g}$ on M by $\tilde{g} = fg$. We denote the subscripts g and $\tilde{g}$ for their corresponding quantities. Then we get

$$\int_M \|R^\nabla\|^2_{\tilde{g}} v_{\tilde{g}} = \int_M f^{(n-4)/2}\|R^\nabla\|^2_g v_g.$$

For the Euler–Lagrange equation,

$$\delta^\nabla_{\tilde{g}} R^\nabla = 0 \quad \text{if and only if } \delta^\nabla_g (f^{(n-4)/2} R^\nabla) = 0,$$

where $\delta^\nabla_{\tilde{g}}$, δ^∇_g are the formal adjoint of d^∇ corresponding to $\tilde{g}$ and g respectively. Moreover, the functional

$$F_p(\nabla) = \frac{1}{2}\int_M (1 + \|R^\nabla\|^2)^{p/2} v_g$$

satisfies the Palais–Smale condition and attains a minimum if $2p > \dim M$. Its Euler–Lagrange equation is given by

$$\delta^\nabla_g ((1 + \|R^\nabla\|^2)^{(p-2)/2} R^\nabla) = 0. \tag{1}$$

In fact, for $A \in \Omega^1(\mathfrak{g}_E)$,

$$\begin{aligned}\frac{d}{dt}|_{t=0} F_p(\nabla + tA) &= \frac{d}{dt}|_{t=0} \int_M (1 + \|R^{\nabla+tA}\|^2_g)^{p/2} v_g \\ &= \frac{p}{2}\int_M (1 + \|R^\nabla\|^2)^{(p-2)/2} < d^\nabla A, R^\nabla >_g v_g.\end{aligned}$$

The equation (1) has a solution ∇ for $2p > \dim M$. For the solution ∇, defining

$$f = (1 + \|R^\nabla\|^2)^{(p-2)/(n-4)}$$

and $\tilde{g} = fg$, we obtain $\delta^\nabla_{\tilde{g}} = 0$, so $\tilde{g}$ and ∇ are the desired ones. □ □

Proposition 6. *The function* $f \mapsto \log f/f^2$ *is a strictly increasing function on the interval* $[1, \sqrt{e})$. *Thus the inverse function* $f = \Psi(y)$ *exists on the interval* $[0, 1/2e)$ *and smooth.*

Proof. In fact, the derivative is

$$\frac{dy}{df} = \frac{1 - 2\log f}{f^3}$$

which is positive on the interval $[1, \sqrt{e})$. □ □

Proposition 7. *Let* (M, g) *be an* n*–dimensional compact Riemannian manifold,* G *a compact Lie group, and* E *a* G*–vector bundle over* M*. Assume that* $n \geq 5$ *and* ∇ *is a Yang–Mills connection. Then for any* $\epsilon > 0$*, there exists a* C^∞ *Riemannian metric* $\tilde{g}$ *on* M *which is homotopic to* g *such that* ∇ *is Yang–Mills connection with respect to* $\tilde{g}$ *and* $\|R^\nabla\|_{\tilde{g}}^2 < \epsilon$.

Proof. For a positive constant C, put $\tilde{g} = Cg$. Then the Yang–Mills equation for $\tilde{g}$ is the same for g. Moreover, since $\|R^\nabla\|_{\tilde{g}}^2 = C^{-2}\|R^\nabla\|_g^2$ and M is compact, we get $\|R^\nabla\|_g^2 < \epsilon$ if C is sufficiently large. □ □

Theorem 8. *Let* (M, g) *be an* n*–dimensional compact Riemannian manifold,* G *a compact Lie group, and* E *a* G*–vector bundle over* M*. Assume that* $n \geq 5$ *and* ∇ *is a Yang–Mills connection. Then there exists a* C^∞ *Riemannian metric on* M *which is conformal to* g *such that* ∇ *is an exponential Yang–Mills connection with respect to* $\tilde{g}$.

Proof. By Proposition 7, we may assume a Yang–Mills connection ∇ satisfies $\|R^\nabla\|^2 < \epsilon < \frac{n-4}{2e}$. For a positive C^∞ function f on M, define $\tilde{g} = f^{-1}g$. Then

$$\delta_g^\nabla R^\nabla = 0 \quad \text{if and only if } \delta_{\tilde{g}}^\nabla (f^{(n-4)/2} R^\nabla) = 0.$$

Since $\|R^\nabla\|_g^2 < \frac{n-4}{2e}$, we can define the function f on M by

$$f = \Psi(\frac{\|R^\nabla\|_g^2}{n-4}) > 0$$

due to Proposition 6. Then it holds that

$$\begin{aligned} f^{(n-4)/2} &= (\exp(\frac{\|R^\nabla\|_g^2}{n-4}))^{(n-4)/2} \\ &= \exp(f^2 \frac{\|R^\nabla\|_g^2}{2}) = \exp(\frac{\|R^\nabla\|_{\tilde{g}}^2}{2}). \end{aligned}$$

Then it holds that

$$\delta_{\tilde{g}}^\nabla (\exp(\frac{\|R^\nabla\|_{\tilde{g}}^2}{2}) R^\nabla) = 0$$

which implies that ∇ is an exponential Yang–Mills connection with respect to $\tilde{g}$.□□

From Theorem 5 and Theorem 8 we obtain the following result.

Theorem 9. *Let (M, g) be an n–dimensional compact Riemannian manifold, G a compact Lie group, and E a G–vector bundle over M. Assume that $n \geq 5$. Then there exists a C^∞ Riemannian metric $\tilde{g}$ on M which is conformal to g and a C^∞ G–connection ∇ on E such that ∇ is an exponential Yang–Mills connection with respect to $\tilde{g}$.*

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