PERTURBATION AND JUMP OF A SEMI-FREDHOLM OPERATOR

Dong Hark Lee and Phil Ung Chung

Abstract. The purpose of the present paper is to derive the perturbations and jumps of semi-Fredholm operators.

1. Introduction

Throughout this paper, we suppose that X is a Banach space and write $BL(X)$ for the set of all bounded linear operators on X.

A linear operator $T \in BL(X)$ is called upper semi-Fredholm if it has closed range with finite dimensional null space, and lower semi-Fredholm if it has closed range with its range of finite codimension.

If T is upper or lower semi-Fredholm, we call it semi-Fredholm.

We shall introduce $T^{\infty}(X) = \cap_{n=1}^{\infty} T^n(X)$ for the hyperrange and $T^{-\infty}(0) = \cup_{n=1}^{\infty} T^{-n}(0)$ for the hyperkernel of $T \in BL(X)$.

$T \in BL(X)$ is hyperexact if $T^{-1}(0) \subseteq T^{\infty}(X)$.

We shall say that T has ascent $\leq k$ if there exists a positive integer k such that $T^{-\infty}(0) = T^{-k}(0)$ and T has descent $\leq k$ if there exists a positive integer k for which $T^{\infty}(X) = T^k(X)$.

The punctured neighborhood theorem says that if $T \in BL(X)$ is semi-Fredholm then there is $\epsilon > 0$ for which $n(T-\lambda)$ and $d(T-\lambda)$ are both constant for $0 < |\lambda| < \epsilon$, $\lambda \in C$, where $n(T) = \dim(T^{-1}(0))$, $d(T) = \operatorname{codim}(T(X))$. We define the jump $j(T)$ of a semi-Fredholm operator $T \in BL(X)$;

$$j(T) \overset{\text{def}}{=} \begin{cases} n(T) - n(T-\lambda) \text{ if } T \text{ is upper semi-Fredholm,} \\ d(T) - d(T-\lambda) \text{ if } T \text{ is lower semi-Fredholm.} \end{cases}$$

T.T. West ([5]) has shown that if $j(T) \neq 0$, then there is the smallest integer t for which $T^{-1}(0) \subseteq T^{t-1}(X)$ but $T^{-1}(0) \nsubseteq T^t(X)$.

The purpose of this paper is to derive the perturbation and jumps of semi-Fredholm operators.

Lemma 1. *Let T, $T_i \in BL(X), i \in N$ be semi-Fredholm operators. Then*

$$\begin{aligned} &(T_1T_2\cdots T_m)^{-1}(0) \subseteq (T_1T_2\cdots T_m)^{\infty}(X) \\ &\quad \textit{iff} \quad (T_1T_2\cdots T_m)^{-\infty}(0) \subseteq (T_1T_2\cdots T_m)^{\infty}(X) \\ &\quad \textit{iff} \quad j(T_1T_2\cdots T_m) = 0 \end{aligned} \tag{1.1}$$

Received December 21, 1995.

1991 Mathematics Subject Classification: 47A53.

Key words and phrases: semi-Fredholm operator, jump, perturbation.

and

$$T^{-m}(0) \subseteq (T^m)^\infty(X) \quad \textit{iff} \quad j(T^m) = 0 \tag{1.2}$$

for each $m \in N$.

proof. Since each T_i is semi-Fredholm, $T_1T_2\cdots T_m$ is also semi-Fredholm. Hence we have

$$\begin{aligned}
(T_1T_2\cdots T_m)^{-\infty}(0) &\subseteq (T_1T_2\cdots T_m)^\infty(X) \\
&\Longleftrightarrow (T_1T_2\cdots T_m)^{-1}(0) \subseteq (T_1T_2\cdots T_m)^n(X) \text{ for each } n \in N \\
&\Longleftrightarrow (T_1T_2\cdots T_m)^{-1}(0) \subseteq (T_1T_2\cdots T_m)^\infty(X).
\end{aligned}$$

and

$$j(T_1T_2\cdots T_m) = 0 \Longleftrightarrow (T_1T_2\cdots T_m)^{-\infty}(0) \subseteq (T_1T_2\cdots T_m)^\infty(X).$$

Using the above argument, we have

$$T^{-m}(0) \subseteq (T^m)^\infty(X) \Longleftrightarrow j(T^m) = 0,$$

for each $m \in N$. □ □

2. Main Results

THEOREM 2. *Let each $T_i \in BL(X), i \in N$ be semi-Fredholmand let k and t be the smallest integers for which , for each $m \in N$,*

$$\begin{aligned}
(T_1T_2\cdots T_m)^{-1}(0) &\cap (T_1T_2\cdots T_m)^\infty(X) \\
&= (T_1T_2\cdots T_m)^{-1}(0) \cap (T_1T_2\cdots T_m)^k(X),
\end{aligned}$$

and

$$(T_1T_2\cdots T_m)^{-1}(0) \subseteq (T_1T_2\cdots T_m)^{t-1}(X),$$

$$(T_1T_2\cdots T_m)^{-1}(0) \nsubseteq (T_1T_2\cdots T_m)^t(X)$$

respectively. If either $k < t$ or $k = t$ and $T_1T_2\cdots T_m$ is hyperexact, then we have

$$d((T_1T_2\cdots T_m)^k) = kd(T_1T_2\cdots T_m) \tag{2.1}$$

and

$$j((T_1T_2\cdots T_m)^k) = kj(T_1T_2\cdots T_m). \tag{2.2}$$

proof. Since each operator of the form $(T_1T_2\cdots T_m)^n,\ n\in N$ is semi-Fredholm, we have

$$X/((T_1T_2\cdots T_m)^n(X)+(T_1T_2\cdots T_m)^{-1}(0)) \cong (T_1T_2\cdots T_m)(X)/(T_1T_2\cdots T_m)^{n+1}(X).$$

In particular, if $k<t$, then

$$(T_1T_2\cdots T_m)^{-1}(0)\subseteq (T_1T_2\cdots T_m)^k(X).$$

Hence we have

$$X/(T_1T_2\cdots T_m)^k(X)\cong (T_1T_2\cdots T_m)(X)/(T_1T_2\cdots T_m)^{k+1}(X).$$

By the inductive steps, we have

$$X/(T_1T_2\cdots T_m)^k(X)\cong k(X/(T_1T_2\cdots T_m)(X)).$$

Since $(T_1\cdots T_m)$, $(T_1\cdots T_m)^k$ are semi-Fredholm, $X/(T_1T_2\cdots T_m)(X)$ and $X/(T_1T_2\cdots T_m)^k(X)$ are finite dimensional normed spaces. Then,

$$d((T_1T_2\cdots T_m)^k)=kd(T_1T_2\cdots T_m).$$

By the duality of $d((T_1T_2\cdots T_m)^k)$, we have

$$n((T_1T_2\cdots T_m)^k)=kn(T_1T_2\cdots T_m).$$

If $k=t$ and $T_1T_2\cdots T_m$ is hyperexact, then

$$(T_1T_2\cdots T_m)^{-1}(0)\subseteq (T_1T_2\cdots T_m)^{k=t}(X)=(T_1T_2\cdots T_m)^{\infty}(X).$$

Since $j(T_1T_2\cdots T_m)=0$ by Lemma 1, we have

$$kn(T_1T_2\cdots T_m)=kn(T_1T_2\cdots T_m-\lambda)=n((T_1T_2\cdots T_m)^k)$$

for sufficiently small λ. Using the punctured neighborhood theorem,

$$n((T_1T_2\cdots T_m-\lambda)^l)-n((T_1T_2\cdots T_m)^l-\mu) = l[n(T_1T_2\cdots T_m)]-l[n(T_1T_2\cdots T_m)]=0$$

for each $l\in N$ and for a sufficiently small $\mu\in C$. Hence

$$n((T_1T_2\cdots T_m)^k)=kn(T_1T_2\cdots T_m),$$

and

$$j((T_1T_2\cdots T_m)^k)=kn(T_1T_2\cdots T_m).$$

By the duality of $n((T_1T_2\cdots T_m)^k)$,

$$d((T_1T_2\cdots T_m)^k)=kd(T_1T_2\cdots T_m).$$

Thus, we have the required results. □ □

Corollary 3. *Let $T \in BL(X)$ be semi-Fredholm and let k and t be the smallest integer such that $T^{-1}(0) \cap T^{\infty}(X) = T^{-1}(0) \cap T^{k}(X)$ and $T^{-1}(0) \subseteq T^{t-1}(X)$ but $T^{-1}(0) \nsubseteq T^{t}(X)$ respectively. If either $k < t$ or $k = t$ and T is hyperexact, then we have*

$$d(T^k) = kd(T), \tag{3.1}$$

$$j(T^k) = kj(T). \tag{3.2.}$$

proof. Using Theorem 2, our assertion can be easily proved. □ □

Theorem 4. *Let S, $T \in BL(X)$ commute.*

(4.1) If T is an upper semi-Fredholm operator with finite ascent, and S is a compact operator with finite ascent, then

$$j(T+S) = n(T+S).$$

(4.2) If T is a lower semi-Fredholm operator with finite descent and S is a compact operator with finite descent, then

$$j(T+S) = d(T+S).$$

(4.3) If $S+T$ is a semi-Fredholm operator with finite ascent and t is the smallest integer for which $(T+S)^{-1}(0) \subseteq (T+S)^{t-1}(X)$ but $(T+S)^{-1}(0) \nsubseteq (T+S)^{t}(X)$ and either $k < t$ or $k = t$ and $T+S$ is hyperexact, then

$$j((T+S)^k) = kj(T+S) = kn(T+S).$$

proof. Suppose that T is upper semi-Fredholm with finite ascent and S is compact operator with finite ascent for (4.1). Then $T+S$ is upper semi-Fredholm and it has finite ascent. If $T+S$ has ascent k, then we have

$$(T+S)^{-1}(0) \cap (T+S)^k(X) = \{0\}.$$

and

$$\dim(T+S-\lambda)^{-1}(0) = \dim((T+S)^{-1}(0) \cap (T+S)^{\infty}(X))$$

for sufficiently small λ. Since

$$(T+S)^{-1}(0) \cap (T+S)^{\infty}(X) = (T+S)^{-1}(0) \cap (T+S)^k(X) = \{0\},$$

we have

$$\dim((T+S-\lambda)^{-1}(0)) = 0.$$

Hence

$$j(T+S) = n(T+S) - n(T+S-\lambda) = n(T+S)$$

for sufficiently small λ. And we have $d(T+S-\lambda) = 0$ for sufficiently small λ([1],[3]). Thus $j(T+S) = d(T+S)$. Let t be the smallest integer for which $(T+S)^{-1}(0) \subseteq (T+S)^{t-1}(X)$ but $(T+S)^{-1}(0) \nsubseteq (T+S)^{t}(X)$ and $k \leq t$ for (4.3). If $T+S$ has finite ascent, then, (4.3) follows at once from Corollary 3 and (4.1). □ □

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Dong Hark Lee
Department of Mathematics Education
Kangwon National University
Chuncheon 200-701, Korea

Phil Ung Chung
Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea